Some Studies in Noncommutative Quantum Field Theories

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CERTIFICATE FROM THE SUPERVISOR

This is to certify that the thesis entitled "**Some Studies in Noncommutative Quantum Field Theories**" submitted by **Sri Sunandan Gangopadhyay** who got his name registered on **19.12.2005** for the award of Ph.D.(Science) degree of West Bengal University of Technology, absolutely based upon his own work under the supervision of **Dr. Biswajit Chakraborty** and that neither this thesis nor any part of it has been submitted for any degree/diploma or any other academic award anywhere before.

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Dedicated

to

My Parents

and

Grand Parents

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Contents

Chapter 1

Introduction and Overview

Noncommutative geometry [1] is presently one of the most important areas of investigation. From a purely mathematical point of view, noncommutative geometry amounts to a program of unification in mathematics under the aegis of the quantum apparatus, i.e. the theory of operators and of C^*-algebras. There has been an explosion of intense research these days by some of the world's leading mathematicians, and a variety of applications starting from the reinterpretation of the phenomenological Standard Model of particle physics as a new spacetime geometry, to the quantum Hall effect, strings, renormalization and more in quantum field theory.

The development of Noncommutative Quantum Field theories historically starts with Heisenberg's observation (in a letter he wrote to Pierls in the late 1930 [2]) on the possibility of introducing uncertatinty relations for coordinates, as a way to avoid singularities of the electron self energy. Pierls made use of these ideas eventually in his work related to the Landau level problem. Heisenberg also commented on this possibility to Pauli who then involved Oppenheimer in the discussion [3]. Finally it was Hartland Snyder, a student of Oppenheimer who first formalised this idea in an artile on *Quantised Space time* [4] entirely devoted to this subject. Almost immediately, C.N. Yang reacted to this paper and published a letter to the Editor of the Physical Review [5] extending Snyder's treatment to the case of curved space (in particular de Sitter space). Then in 1948, Moyal addressed the problem using Wigner phase-space distribution functions and he introduced what is known as the Moyal star product, a noncommutative associative product, in order to discuss the mathematical structure of quantum mechanics [6]. For a simple classical system like a particle moving on a real line, the construction of the star product can be motivated by considering the set of Weyl ordered phase-space operators and its isomorphism to the set of classical phase-space functions. (See [7] for a review.) This result has also been shown later through a geometric approach by Berezin [8], Batalin and Tyutin [9].

The contemporary success of the renormalisation program shadowed these ideas for a while. However, the ideas of noncommutative geometry were once again revived in the 1980's by the mathematicians Connes, Woronowicz and Drinfel'd, who generalised the notion of a differential structure to the noncommutative setting [1]. Just as it is possible to give many differential structures to a given topological space, it is possible to define many differential calculi over a given algebra. Along with the introduction of a generalized integral [10], this permits one in principle to define the action of a Yang-Mills field on a large class of noncommutative geometries.

More concrete evidence for spacetime noncommutativity came from string theory, which at present is arguably the most promising candidate for a quantum theory of gravity. Strings having a finite intrinsic length scale l_s, can be used as probes of short distance structure. Hence, distances smaller than l_s are not possible to observe. In fact, based on the analysis of very high-energy string scattering amplitudes [11, 12, 13], string-modified Heisenberg uncertainty relations have been postulated in the form:

$$\Delta x = \frac{\hbar}{2}\left(\frac{1}{\Delta p} + l_s^2 \Delta p\right). \tag{1.1}$$

It is easy to see that one recovers the usual quantum mechanical result in the limit $l_s \to 0$.

The seminal paper of Seiberg and Witten [14] identified limits in which the entire string dynamics can be described in terms of a minimally coupled (supersymmetric) gauge theory on a noncommutative space. Their analysis leads to an equivalence between ordinary gauge fields and noncommutative gauge fields, realized by a change of variables that can be described explicitly. This change of variables (commonly known as the Seiberg-Witten (SW) map in the literature) is checked by comparing the ordinary Dirac-Born-Infeld theory with its noncommutative counterpart.

The central theme of this thesis is to study some aspects of noncommutative quantum mechanics and noncommutative quantum field theory. We explore how noncommutative structures can emerge and study the consequences of such structures in various physical models. The outline of this thesis is as follows.

• We present a review of noncommutative quantum mechanics in chapter 2 where we discuss the procedure of Weyl quantization which is an useful technique for translating an ordinary field theory into a noncommutative one. The Weyl operators are introduced and the Weyl-Wigner correspondence is derived. We then move on to present an alternative route to the star product formalism following [7].

• In chapter 3, as a "warm-up exercise", we demonstrate how noncommuting structures can be obtained in the first place by exploiting the reparametrization symmetry of particle models. Studies has been going on for some time in this direction and it has been observed that an important role in this context is played by change of variables which provide a map among the commutative and noncommutative structures. However, a precise underying principle on which such maps are based was found to be missing. We have made a thorough study giving a systematic formulation of such maps, where they are essentially gauge/reparametrization transformations.

• As we have mentioned earlier, the SW map has played a central role in the analysis of noncommutative quantum field theories as it provides a map from the noncommutative to the commutative space, while preserving the gauge invariance. On the other hand, issues related to the violation of Lorentz symmetry in noncommutative relativistic systems have become important and studies have been done using noncommutative variables or with their equivalent commutative counterpart obtained by SW map.

In chapter 4, we have carried out investigations in this line by constructing an effective $U(1)$ gauge invariant theory for a noncommutative nonrelativistic model, where the Schrödinger field is coupled to a $U(1)_\star$ gauge field in $2+1$-dimensions, using the first order SW map. We study how this effective theory can be cast in the form of usual Schrödinger action with interaction terms of noncommutative origin. We then explore the Galilean symmetry of the model in details and observe a violation of the above symmetry in our model. This violation is shown to be a noncommutative effect. As an application of our effective model, we have also computed the Hall conductivity and find that there is no correction due to noncommutativity.

• In chapter 5, we go through a detailed study of noncommutative quantum mechanics. Here we carry out the construction of a one parameter family of interacting noncommuting physically equivalent Hamiltonians (i.e. Hamiltonians having the same spectrum). We have been able to perform this construction exactly (to all orders in the noncommutative parameter θ) and analytically in two dimensions for a free particle and a harmonic oscillator in the presence of a constant magnetic field. We then investigate the implications of the SW map in this context in details. Finally, we work out an approximate duality between interacting commutative and weakly interacting noncommutative Hamiltonians for harmonic oscillator potentials.

• In chapter 6, we take up the quantum Hall system which has been an important area of application of two dimensional noncommutative quantum systems. Here, we discuss the role that interactions play in the noncommutative structure that arises when the relative coordinates of two interacting particles are projected onto the lowest Landau level. It is shown that the interactions in general renormalize the noncommutative parameter away from the non-interacting value $\frac{1}{B}$. The effective noncommutative parameter is in general also angular momentum dependent. An heuristic argument, based on the noncommutative coordinates, is given to find the filling fractions at incompressibilty, which are in general renormalized by the interactions, and the results are consistent with known results in the case of singular magnetic fields.

• The twist approach to noncommutative quantum field theory has recently gained a lot of popularity. As mentioned earlier, breaking of Lorentz invariance following from the choice of a particular noncommutative matrix θ have become important in noncommutative relativistic systems. The twist approach was proposed as a way to circumvent this problem. It was triggered by the realization that it is possible to twist the coproduct of the universal envelope $U(\mathcal{P})$ of the Poincaré algebra, which is a Hopf algebra, such that it is compatible with the $\star$-product. Two interesting consequences follow from the twisted implementation of the Poincaré group. Firstly, the IR/UV mixing is no longer there which implies that the high and low energy sectors decouple, in contrast to the untwisted formulation. The second important consequence is an apparent violation of Pauli's exclusion principle.

In chapter 7, we show the twisted Galilean invariance of the noncommutative parameter, even in presence of spacetime noncommutativity. The deformed algebra of the Schrödinger field is then obtained in configuration and momentum space by studying the action of the twisted Galilean group on the nonrelativistic limit of the Klein-Gordon field and can be extended in a straightforward manner for the Dirac field also. This deformed algebra is used to compute the two particle correlation function to study the possible extent to which the previously proposed violation of the Pauli principle may impact at low energies. It is concluded that any possible effect is probably well beyond detection at current energies.

- Finally, we end up with conclusions in chapter 8.

This thesis is based on the following publications.

1. Seiberg-Witten map and Galilean symmetry violation in a noncommutative planar system [15]

   B. Chakraborty, **S. Gangopadhyay**, A. Saha,

   Phys. Rev. D **70** (2004) 107707.

2. Noncommutativity and reparametrization symmetry [16]

   R. Banerjee, B. Chakraborty, **S. Gangopadhyay**,

   J. Phys. A. **38** (2005) 957.

3. Dual families of noncommutative quantum systems [18]

   F.G. Scholtz, B. Chakraborty, **S. Gangopadhyay**, A.G. Hazra,

   Phys. Rev. D **71** (2005) 085005.

4. Interactions and noncommutativity in quantum Hall systems [19]

   F.G. Scholtz, B. Chakraborty, **S. Gangopadhyay**, J. Govaerts,

   J. Phys. A **38** (2005) 9849.

5. Twisted Galilean symmetry and the Pauli principle at low energies [20]

   B. Chakraborty, **S. Gangopadhyay**, A.G. Hazra, F.G. Scholtz,

   J. Phys. A **39** (2006) 9557.

6. Lie algebraic noncommuting structures from reparametrization symmetry [17]

   S. Gangopadhyay,

   J. Math. Phys **48** (2007) 052302.

Chapter 2

Review of Noncommutative Quantum Mechanics and Introduction to Star product

2.1 Weyl quantization and Groenewold-Moyal product

The idea behind spacetime noncommutativity is very much inspired by the foundations of quantum mechanics. Within the framework of canonical quantization, Weyl introduced an elegant prescription for associating a quantum operator to a classical function of the phase-space variables [21]. This programme leads to a deep conceptual revolution because the emphasis on group-theoretical methods provides a scheme where Weyl systems can be considered in the first place [22] and classical mechanics is eventually recovered. Further, this technique provides a systematic way to describe noncommutative spaces in general and to study field theories defined thereon. In this section we shall introduce this formalism which will play a central role in most of our subsequent analysis. It is also worthwhile to mention that Weyl quantization works for very general type of commutation relations[1].

[1] In the following section, we have drawn freely from [23]. Some of the intermediate steps in the derivation of the key results has been worked out in details.

2.1.1 Weyl Operators

Let us consider the commutative algebra of (possibly complex-valued) functions on D-dimensional Euclidean space $\mathcal{R}^D$, with the usual pointwise multiplication of functions defined as the product. We will assume that all fields defined on $\mathcal{R}^D$ live in an appropriate Schwartz space of functions of sufficiently rapid decrease at infinity [24], i.e. those functions whose derivatives vanish at infinity in both position and momentum space to arbitrary order. This condition can be characterized, for example, by the requirements

$$\sup_x \left(1+|x|^2\right)^{k+n_1+\ldots+n_D} \left|\partial_1^{n_1}\cdots\partial_D^{n_D} f(x)\right|^2 \;<\; \infty \tag{2.1}$$

for every set of integers $k, n_i \in \mathcal{Z}_+$, where $\partial_i = \partial/\partial x^i$. In that case, the algebra of functions may be given the structure of a Banach space by defining the L^∞-norm

$$\|f\|_\infty = \sup_x \left|f(x)\right| \;. \tag{2.2}$$

The Schwartz condition also implies that any function $f(x)$ may be described by its Fourier transform

$$\tilde{f}(k) = \int_{-\infty}^{+\infty} d^D x \; f(x) e^{-ik_i x^i} \tag{2.3}$$

with $\tilde{f}(-k) = \tilde{f}(k)^*$ whenever $f(x)$ is real-valued. We now define a noncommutative space by replacing the local coordinates x^i of $\mathcal{R}^D$ by Hermitian operators $\hat{x}^i$ obeying the commutation relations:

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}. \tag{2.4}$$

The noncommutative algebra of operators is then generated by $\hat{x}^i$. A one-to-one correspondence between the algebra of fields on $\mathcal{R}^D$ and this ring of operators is provided by Weyl quantization, and it may be thought of as an analog of the operator-state correspondence of local quantum field theory. Given the function $f(x)$ and its corresponding Fourier coefficients (2.3), one can introduce its *Weyl symbol* by

$$\hat{\mathcal{W}}[f] = \int_{-\infty}^{+\infty} \frac{d^D k}{(2\pi)^D} \tilde{f}(k) e^{ik_i \hat{x}^i} \tag{2.5}$$

where we have chosen the symmetric Weyl operator ordering prescription.

For example, choosing $f(x) = e^{ik_ix^i}$, eq.(s) (2.3) and (2.5) leads to:

$$\begin{aligned}
\hat{\mathcal{W}}[e^{ik_ix^i}] &= \int_{-\infty}^{+\infty} \frac{d^Dk'd^Dy}{(2\pi)^D} e^{ik'_i(\hat{x}^i - y^i)} e^{ik_iy^i} \\
&= \int_{-\infty}^{+\infty} d^Dk' e^{ik'_i\hat{x}^i} \delta^{(D)}(k_i - k'_i) \\
&= e^{ik_i\hat{x}^i}.
\end{aligned} \tag{2.6}$$

Note that the Weyl operator $\hat{\mathcal{W}}[f]$ is Hermitian if $f(x)$ is real-valued. Using eq.(2.3), one can write eq.(2.5) in terms of an explicit map $\hat{\Delta}(x)$ between operators and fields to get

$$\hat{\mathcal{W}}[f] = \int_{-\infty}^{+\infty} d^Dx \; f(x)\,\hat{\Delta}(x) \tag{2.7}$$

where,

$$\hat{\Delta}(x) = \int_{-\infty}^{+\infty} \frac{d^Dk}{(2\pi)^D} e^{ik_i(\hat{x}^i - x^i)}. \tag{2.8}$$

The operator (2.8) is Hermitian, $\hat{\Delta}(x)^\dagger = \hat{\Delta}(x)$, and it describes a mixed basis for operators and fields on spacetime. In this way we may interpret the field $f(x)$ as the coordinate space representation of the Weyl operator $\hat{\mathcal{W}}[f]$. Note that in the commutative case $\theta^{ij} = 0$, the map (2.8) reduces trivially to a delta-function $\delta^D(\hat{x} - x)$ and $\hat{\mathcal{W}}[f]|_{\theta=0} = f(\hat{x})$. But generally, by the Baker-Campbell-Hausdorff (BCH) formula, for $\theta^{ij} \neq 0$ it is a highly non-trivial field operator. To proceed further, we now introduce "derivatives" of operators through an anti-Hermitian linear derivation $\hat{\partial}_i$ defined by the commutation relations

$$\left[\hat{\partial}_i\,,\,\hat{x}^j\right] = \delta_i{}^j \qquad , \qquad \left[\hat{\partial}_i\,,\,\hat{\partial}_j\right] = 0. \tag{2.9}$$

Then after a little algebra, it is straightforward to show that

$$\left[\hat{\partial}_i\,,\,\hat{\Delta}(x)\right] = -\partial_i\,\hat{\Delta}(x) \tag{2.10}$$

which upon integration by parts in eq.(2.7) leads to

$$\left[\hat{\partial}_i\,,\,\hat{\mathcal{W}}[f]\right] = \int_{-\infty}^{+\infty} d^Dx \; \partial_if(x)\,\hat{\Delta}(x) = \hat{\mathcal{W}}[\partial_if]. \tag{2.11}$$

Now using eq.(s) (2.8), (2.9), (2.10) and the BCH-formula

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]},\;\; [\hat{A},\hat{B}] = c \tag{2.12}$$

(where, c is a number) we find that the computation of $e^{v^i\hat{\partial}_i}\,\hat{\Delta}(x)\,e^{-v^i\hat{\partial}_i}$ leads to:

$$\begin{aligned} e^{v^i\hat{\partial}_i}\,\hat{\Delta}(x)\,e^{-v^i\hat{\partial}_i} &= \int_{-\infty}^{+\infty}\frac{d^Dk}{(2\pi)^D}e^{v^i\hat{\partial}_i}e^{ik_i\hat{x}^i}e^{-v^j\hat{\partial}_j}e^{-ik_ix^i} \\ &= \int_{-\infty}^{+\infty}\frac{d^Dk}{(2\pi)^D}e^{ik_i[\hat{x}^i-(x^i-v^i)]} \\ &= \hat{\Delta}(x+v). \end{aligned} \tag{2.13}$$

Eq.(2.13) implies that translation generators can be represented by unitary operators $e^{v^i\hat{\partial}_i}$ ($v \in \mathcal{R}^D$). The property (2.13) also implies that any cyclic trace tr defined on the algebra of Weyl operators has the feature that $\hat{\Delta}(x)$ is independent of $x \in \mathcal{R}^D$. From eq.(2.7) it follows that the trace tr is uniquely given by an integration over spacetime

$$tr\ \hat{\mathcal{W}}[f] = \int_{-\infty}^{+\infty} d^Dx\ f(x) \tag{2.14}$$

where we have chosen the normalization $\hat{\Delta}(x) = 1$. In this sense, the operator trace tr is equivalent to integration over the noncommuting coordinates $\hat{x}^i$.

With the above results at our hands, we compute the products of operators $\hat{\Delta}(x)$ at distinct points as follows. To begin with, let us observe that the BCH-formula (2.12) yields:

$$e^{ik_i\hat{x}^i}\ e^{ik'_j\hat{x}^j} = e^{-\frac{i}{2}\theta^{ij}k_ik'_j}e^{i(k+k')_i\hat{x}^i}. \tag{2.15}$$

This along with eq.(2.8), leads to:

$$\begin{aligned} \hat{\Delta}(x)\hat{\Delta}(y) &= \int_{-\infty}^{+\infty}\frac{d^Dk}{(2\pi)^D}\frac{d^Dk'}{(2\pi)^D}e^{ik_i(\hat{x}^i-x^i)}e^{ik'_j(\hat{x}^j-y^j)} \\ &= \int_{-\infty}^{+\infty}\frac{d^Dk}{(2\pi)^D}\frac{d^Dk'}{(2\pi)^D}e^{i(k_i+k'_i)\hat{x}^i}e^{-\frac{i}{2}\theta^{ij}k_ik'_j}e^{-ik_ix^i-ik'_iy^i} \\ &= \int_{-\infty}^{+\infty}\frac{d^Dkd^Dk'}{(2\pi)^{2D}}\left[\int_{-\infty}^{+\infty}d^Dze^{i(k_i+k'_i)z^i}\hat{\Delta}(z)\right]e^{-\frac{i}{2}\theta^{ij}k_ik'_j}e^{-ik_ix^i-k'_iy^i} \\ &= \int_{-\infty}^{+\infty}\frac{d^Dz}{(2\pi)^{2D}}\hat{\Delta}(z)\int_{-\infty}^{+\infty}d^Dke^{ik_i(z^i-x^i)}\int_{-\infty}^{+\infty}d^Dk'e^{ik'_j(z^j-y^j)}e^{-\frac{i}{2}\theta^{ij}k_ik'_j} \\ &= \int_{-\infty}^{+\infty}\frac{d^Dz}{(2\pi)^D}\hat{\Delta}(z)\int_{-\infty}^{+\infty}d^Dke^{ik_i(z^i-x^i)}\delta^{(D)}(\frac{1}{2}k_i\theta^{ij}-a^j) \end{aligned} \tag{2.16}$$

where in the third line, we have used

$$e^{ik_i\hat{x}^i} = \int_{-\infty}^{+\infty}d^Dz\hat{\Delta}(z)e^{ik_iz^i}. \tag{2.17}$$

If θ is an invertible matrix (this necessarily requires that the spacetime dimension D be even), then the delta function integration over the momentum k in eq.(2.16) can be explicitly carried out to get

$$\hat{\Delta}(x)\,\hat{\Delta}(y) = \frac{1}{\pi^D|\det\theta|}\int_{-\infty}^{+\infty} d^Dz\;\hat{\Delta}(z)\;\exp^{-2i(\theta^{-1})_{ij}(x-z)^i(y-z)^j}\;. \tag{2.18}$$

It follows from eq.(2.18), by the use of the trace normalization and the antisymmetry of θ^{-1}, that the operators $\hat{\Delta}(x)$ (for $x \in \mathcal{R}^D$) form an orthonormal set

$$tr\,\left(\hat{\Delta}(x)\,\hat{\Delta}(y)\right) = \delta^D(x-y)\;. \tag{2.19}$$

This, along with eq.(2.7), implies that the transformation $f(x) \stackrel{\hat{\Delta}(x)}{\longmapsto} \hat{\mathcal{W}}[f]$ is invertible with inverse given by:

$$f(x) = tr\,\left(\hat{\mathcal{W}}[f]\,\hat{\Delta}(x)\right)\;. \tag{2.20}$$

The function $f(x)$ obtained in this way from a quantum operator is usually called a *Wigner distribution function* [25]. Therefore, the map $\hat{\Delta}(x)$ provides a one-to-one correspondence between Wigner fields and Weyl operators[2]. This is usually referred in the literature as the *Weyl-Wigner correspondence*.

2.1.2 The Star-Product

We are now in a position to derive the form of the star product. We begin by considering the product of two Weyl operators $\hat{\mathcal{W}}[f]$ and $\hat{\mathcal{W}}[g]$ corresponding to functions $f(x)$ and $g(x)$. From eq.(s) (2.7), (2.18) and (2.19) it follows that the coordinate space representation of their product can be written (for invertible θ) as

$$\begin{aligned}
tr\,\left(\hat{\mathcal{W}}[f]\,\hat{\mathcal{W}}[g]\,\hat{\Delta}(x)\right) &= tr\left[\int_{-\infty}^{+\infty} d^Dy\; d^Dz\;\hat{\Delta}(y)\hat{\Delta}(z)f(y)\,g(z)\hat{\Delta}(x)\right] \\
&= \int_{-\infty}^{+\infty}\frac{d^Dy\;d^Dz\;d^Dw}{\pi^D|det\theta|}f(y)\,g(z)\exp^{-2i(\theta^{-1})_{ij}(y-w)^i(z-w)^j}\,tr(\hat{\Delta}(w)\hat{\Delta}(x)) \\
&= \frac{1}{\pi^D|\det\theta|}\int_{-\infty}^{+\infty} d^Dy\;d^Dz\;f(y)\,g(z)\;\exp^{-2i(\theta^{-1})_{ij}(x-y)^i(x-z)^j}
\end{aligned}$$

[2]An explicit formula for eq.(2.8) in terms of parity operators can be found in [26, 27].

$$
\begin{aligned}
&= \int_{-\infty}^{+\infty} \frac{d^D k \; d^D k'}{(2\pi)^D} \tilde{f}(k)\tilde{g}(k') e^{ik_i x^i} e^{-\frac{i}{2}\theta^{ij} k_i k'_j} e^{ik'_j x^j} \\
&= f(x) e^{\frac{i}{2}\overleftarrow{\partial}_i \theta^{ij} \overrightarrow{\partial}_j} g(x) \\
&\equiv (f \star g)(x)
\end{aligned}
\tag{2.21}
$$

where we have used eq.(s) (2.3), (2.5) and (2.15) and introduced the *Groenewold-Moyal star-product* [6]. On the other hand

$$
\begin{aligned}
tr\left(\hat{\mathcal{W}}[f\star g]\hat{\Delta}(x)\right) &= tr\left[\int_{-\infty}^{+\infty} d^D z \hat{\Delta}(z)(f\star g)(z)\hat{\Delta}(x)\right] \\
&= \int_{-\infty}^{+\infty} d^D z (f\star g)(z)\; tr\left(\hat{\Delta}(z)\hat{\Delta}(x)\right) \\
&= \int_{-\infty}^{+\infty} d^D z (f\star g)(z)\delta^{(D)}(z-x) \\
&= (f\star g)(x).
\end{aligned}
\tag{2.22}
$$

From eq.(s) (2.21) and (2.22), we finally obtain the celebrated *Weyl-Wigner correspondence*

$$
\hat{\mathcal{W}}[f]\,\hat{\mathcal{W}}[g] = \hat{\mathcal{W}}[f\star g]. \tag{2.23}
$$

2.2 Another approach to star product formalism

In this section we present an alternative approach to the basic ideas of star product formalism essentially following [7]. We consider the case of a particle moving on a real line $\mathcal{R}^1$ as an illustrative example. Clearly the classical phase-space (x,p) is the two dimensional space $\mathcal{R}^2$. An arbitrary phase-space function $f(x,p)$ can be written as

$$
\begin{aligned}
f(x,p) &= \int_{-\infty}^{+\infty} dx' dp' \delta(x-x')\delta(p-p') f(x',p') \\
&= \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty} dx' dp' d\tau d\sigma e^{i[\tau(x-x')+\sigma(p-p')]} f(x',p')
\end{aligned}
\tag{2.24}
$$

where the integral representation

$$
\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\tau e^{i\tau(x-x')} \tag{2.25}
$$

of the Dirac delta function $\delta(x-x')$ and a similar representation for $\delta(p-p')$ are used. At the quantum level, the operator analogues $\hat{x}, \hat{p}$ of x, p obey the Heisenberg-Weyl Lie algebra

$$[\hat{x}, \hat{p}] = i\hbar \quad , \quad [\hat{x}, \hat{x}] = 0 \quad , \quad [\hat{p}, \hat{p}] = 0 \tag{2.26}$$

and $\exp[i(\tau\hat{x} + \sigma\hat{p})]$ is a particular element of the corresponding Lie group.

As we have seen in the earlier section, Weyl's prescription [21] for arriving at the operator $\hat{f}(\hat{x}, \hat{p})$ corresponding to the phase-space function $f(x,p)$ (taken to have a polynomial form) consists of rewriting eq.(2.24) with the replacements $x \to \hat{x}, p \to \hat{p}$ to get:

$$\hat{f}(\hat{x}, \hat{p}) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} dx' dp' d\tau d\sigma e^{i[\tau(\hat{x}-x') + \sigma(\hat{p}-p')]} f(x', p'). \tag{2.27}$$

An equivalent prescription due to Batalin and Tyutin [9] is to define[3]

$$\hat{f}(\hat{x}, \hat{p}) = e^{[\hat{x}\partial_x + \hat{p}\partial_p]} f(x,p)|_{x=p=0} \ . \tag{2.28}$$

We however continue with the prescription (2.27) for the time being.

Now using the mapping (2.27), one can obtain the phase-space function (also called the classical kernel) of the operator product $\hat{f}\hat{g}$ of two phase-space operators $\hat{f}$ and $\hat{g}$ from the corresponding kernels f and g respectively. For that one has to express $\hat{g}(\hat{x}, \hat{p})$ just in the manner of $\hat{f}$ in eq.(2.27). One can then write

$$\begin{aligned} \hat{f}\hat{g} &= \frac{1}{(2\pi)^4} \int_{-\infty}^{+\infty} d\xi d\eta d\xi' d\eta' dx' dx'' dp' dp'' f(x', p') g(x'', p'') \\ &\qquad \times \exp i(\xi(\hat{p} - p') + \eta(\hat{x} - x')) \exp i(\xi'(\hat{p} - p'') + \eta'(\hat{x} - x'')) \\ &= \frac{1}{(2\pi)^4} \int_{-\infty}^{+\infty} d\xi \ d\eta \ d\xi' \ d\eta' \ dx' \ dx'' \ dp' \ dp'' \ f(x', p') g(x'', p'') \exp i\left((\xi + \xi')\hat{p} + (\eta + \eta')\hat{x}\right) \\ &\qquad \times \ \exp\left(-\xi p' - \eta x' - \xi' p'' - \eta' x'' + \frac{\hbar}{2}(\xi\eta' - \eta\xi')\right). \end{aligned} \tag{2.29}$$

Changing integration variables to

$$\xi' \equiv \frac{2}{\hbar}(x - x'), \quad \xi \equiv \tau - \frac{2}{\hbar}(x - x'), \quad \eta' \equiv \frac{2}{\hbar}(p' - p), \quad \eta \equiv \sigma - \frac{2}{\hbar}(p' - p) \tag{2.30}$$

[3]The choice of the origin $x = p = 0$ for evaluating $\hat{f}(\hat{x}, \hat{p})$ is not mandatory. The operator $\hat{f}(\hat{x}, \hat{p})$, evaluated at different points, are in fact related by canonical transformations [9].

reduces the above integral to

$$\hat{f}\hat{g} = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} d\tau d\sigma dx\ dp\ \ \exp i \left(\tau(\hat{p}-p) + \sigma(\hat{x}-x)\right)$$

$$\times \left\{ \int_{-\infty}^{+\infty} dp' dp'' dx' dx''\ f(x',p')\ g(x'',p'')\ \left[\frac{1}{(\pi\hbar)^2} \exp\left(\frac{-2i}{\hbar} \left(p(x'-x'') + p'(x''-x) + p''(x-x')\right)\right)\right]\right\}. \tag{2.31}$$

We consider now the exponential inside the square bracket in the above equation:

$$\begin{aligned}
&\frac{1}{(\pi\hbar)^2} \exp\left(\frac{-2i}{\hbar} \left(p(x'-x'') + p'(x''-x) + p''(x-x')\right)\right) \\
&= \frac{1}{(\pi\hbar)^2} \exp\left(\frac{i}{\hbar} \left(-2(p'-p)(x''-x) + 2(x'-x)(p''-p)\right)\right) \\
&= \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} d\lambda\ d\mu\ \delta(x'-x-\frac{\mu\hbar}{2})\delta(p'-p+\frac{\lambda\hbar}{2}) \exp\left(i\left(\lambda(x''-x) + \mu(p''-p)\right)\right) \\
&= \frac{1}{(2\pi)^4} \int_{-\infty}^{+\infty} d\lambda\ d\mu\ d\alpha\ d\beta\ \ \exp i[\alpha(x'-x) + \beta(p'-p)] \exp \frac{i\hbar}{2} (\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x) \\
&\qquad \times \exp\left(i\left(\lambda(x''-x) + \mu(p''-p)\right)\right)
\end{aligned}$$

where the representation (2.25) is used. With the aid of the above relation, the integral in the curly bracket in eq.(2.31) can be written as

$$\begin{aligned}
&\frac{1}{(2\pi)^4} \int_{-\infty}^{+\infty} d\lambda d\mu d\alpha d\beta dx' dp' dx'' dp'' \exp i[\alpha(x'-x) + \beta(p'-p)] \exp i\hbar(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)/2 \\
&\qquad \times \exp\left(i\left(\lambda(x''-x) + \mu(p''-p)\right)\right)\ f(x',p')\ g(x'',p'') \\
&\qquad = f(x,p)\ e^{\frac{i\hbar}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)/2}\ g(x,p).
\end{aligned} \tag{2.32}$$

Hence the composition rule is given by:

$$\hat{f}\hat{g} = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} d\tau d\sigma dx dp \exp[i\left(\tau(\hat{p}-p) + \sigma(\hat{x}-x)\right)](f \star g)(x,p) \tag{2.33}$$

where the $\star$ product is defined as

$$f(x,p) \star g(x,p) \equiv f(x,p)\ e^{\frac{i\hbar}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)}\ g(x,p). \tag{2.34}$$

Thus the fact that $f(x,p) \star g(x,p)$ is the the classical kernel of $\hat{f}\hat{g}$ has been established [28].

Chapter 3

Noncommutative structures from Reparametrization symmetry

In the previous chapter, we have discussed the basic formalism on which the foundations of noncommutative quantum field theory is based. Now we shall first demonstrate how noncommutative space-space or (spacetime) structures can arise from reparametrization symmetry of particle models. Investigations in this line has been carried out recently in simple particle models and it has been observed that noncommutative structures emerge by suitable change of variables providing a map among the commutative and noncommutative structures [29], [30], [31], [32]. However, these studies lack a precise underlying principle on which such maps are based. One of the motives of our work is to provide a systematic formulation of such maps [16]. In the models that we discuss here, these maps are essentially gauge/reparametrization transformations.

To start with, we first consider the case of the nonrelativistic (NR) free particle in details. Interestingly, even though the model does not have any natural reparametrization symmetry, we can introduce it by hand and then exploit it in order to reveal the various noncommuting structures. As other examples, we consider the free relativistic particle as well as its interaction with a background electromagnetic field.

The methodology that we adopt is to utilize the reparametrization invariance of the model

to find a non-standard gauge in which the spacetime and/or space-space coordinates become noncommuting. We also show that the variable redefinition relating the nonstandard and standard gauges is a gauge transformation.
The structure of the angular momentum operator is then studied in some details. A gauge independent expression is obtained, which therefore does not require any central extension in the non-standard gauge.

Another important point to note is that the structures that we obtain are Lie-algebraic in the case of the NR free particle, but not so in its relativistic counterpart. However, in [17] we have shown that there exists some special choice of the reparametrization parameter for which one can obtain noncommuting space-space structures falling in the Lie-algebraic category even in the relativistic case. We emphasize that these Lie-algebraic structures may be useful in giving explicit forms of the star products and SW maps (discussed in [33]) by reading off the structure constants of the algebra.
Moreover, there exists solutions of ϵ for which the noncommutativity between spatial coordinates vanish, but the spacetime algebra still remains noncommutative.

Finally, there are two appendices in this chapter. In appendix A, we demonstrate the connection between Dirac brackets (DB) in the axial and radiation gauges using suitable gauge transformations. In appendix B, we show using the symplectic formalism, the connection between integral curves and the equations of motion in the time reparametrized version. This also indicates how constraints come into picture naturally in the time-reparametrized formulation.

3.1 Particle models

Let us start from the action for a point particle in classical mechanics

$$S[x(t)] = \int_{t_1}^{t_2} dt L\left(x, \dot{x}\right) \quad ; \quad \dot{x} = \frac{dx}{d\tau} \ . \tag{3.1}$$

It is easy to rewrite the above form of the action in a time-reparametrized invariant form by elevating the status of time t to that of an additional variable, along with x, in the configuration space as

$$S[x(\tau), t(\tau)] = \int_{\tau_1}^{\tau_2} d\tau \dot{t} L\left(x, \frac{\dot{x}}{\dot{t}}\right) = \int_{\tau_1}^{\tau_2} d\tau L_\tau \left(x, \dot{x}, t, \dot{t}\right) \tag{3.2}$$

where,

$$L_\tau(x, \dot{x}, t, \dot{t}) = \dot{t} L\left(x, \frac{\dot{x}}{\dot{t}}\right) \quad ; \quad \dot{t} = \frac{dt}{d\tau} \tag{3.3}$$

and τ is the new evolution parameter which can be taken to be an arbitrary monotonically increasing function of time t. The canonically conjugate momenta corresponding to the coordinates t and x are given by:

$$\begin{aligned} p_t &= \frac{\partial L_\tau}{\partial \dot{t}} = L\left(x, \frac{\dot{x}}{\dot{t}}\right) + \dot{t}\frac{\partial L\left(x, \frac{\dot{x}}{\dot{t}}\right)}{\partial \dot{t}} \\ &= L\left(x, \frac{dx}{dt}\right) - \frac{dx}{dt}\frac{\partial L(x, dx/dt)}{\partial (dx/dt)} = -H \end{aligned} \tag{3.4}$$

$$p_x = \frac{\partial L_\tau}{\partial \dot{x}} \ . \tag{3.5}$$

Now for a time-reparametrized theory, the canonical Hamiltonian (using eq.(s) (3.4, 3.5)) vanishes:

$$H_\tau = p_t \dot{t} + p_x \dot{x} - L_\tau = \dot{t}(H + p_t) = 0. \tag{3.6}$$

As a particular case of eq.(3.1), we start from the action of a free NR particle in one dimension

$$S = \int dt \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 \ . \tag{3.7}$$

We rewrite the above form of the action in a time-reparametrized invariant form as in eq.(3.2):

$$S = \int d\tau L_\tau(x, \dot{x}, t, \dot{t}) \tag{3.8}$$

where,

$$L_\tau(x, \dot{x}, t, \dot{t}) = \frac{m}{2}\frac{\dot{x}^2}{\dot{t}} \qquad ; \qquad \dot{x} = \frac{dx}{d\tau} \quad , \quad \dot{t} = \frac{dt}{d\tau} \ . \tag{3.9}$$

Now the canonical momenta corresponding to the coordinates t and x are given by

$$p_t = \frac{\partial L_\tau}{\partial \dot{t}} = -\frac{m\dot{x}^2}{2\dot{t}^2} \tag{3.10}$$

$$p_x = \frac{\partial L_\tau}{\partial \dot{x}} = \frac{m\dot{x}}{\dot{t}} \tag{3.11}$$

which satisfy the standard canonical Poisson bracket (PB) relations

$$\{x, p_x\} = \{t, p_t\} = 1 \quad ; \quad \{x, x\} = \{p_x, p_x\} = \{t, t\} = \{p_t, p_t\} = 0 \ . \tag{3.12}$$

The fact that the canonical Hamiltonian vanishes for a time-reparametrized theory can be easily checked using eq.(s) (3.10) and (3.11). Also, the primary constraint in the theory, obtained from eq.(s) (3.10, 3.11) is given by

$$\phi_1 = p_x^2 + 2mp_t \approx 0 \tag{3.13}$$

where ≈ 0 implies equality in the "weak" sense [34]. Clearly the spacetime coordinate $x^\mu(\tau)$, ($\mu = 0, 1; x^0 = t, x^1 = x$), transforms as a scalar under reparametrization:

$$\begin{aligned} \tau \rightarrow \tau^{'} &= \tau^{'}(\tau) \\ x^\mu(\tau) \rightarrow x^{'\mu}(\tau^{'}) &= x^\mu(\tau) \ . \end{aligned} \tag{3.14}$$

Hence, the infinitesimal change in the spacetime coordinate ($\delta x^\mu(\tau)$) under an infinitesimal reparametrization transformation ($\tau^{'} = \tau - \epsilon$), is given by

$$\delta x^\mu(\tau) = x^{'\mu}(\tau) - x^\mu(\tau) = \epsilon \frac{dx^\mu}{d\tau} \ . \tag{3.15}$$

Now we proceed to find the generator of this reparametrization transformation. To do this, we first write the variation in the Lagrangian L_τ (3.9) under the transformation (3.15) as a total derivative:

$$\delta L_\tau = \frac{dB}{d\tau} \qquad ; \qquad B = \frac{m\epsilon}{2} \frac{\dot{x}^2}{\dot{t}} \ . \tag{3.16}$$

The usual Noether's prescription can then be used to obtain the generator G as

$$G = p_t \delta t + p_x \delta x - B = \frac{\epsilon \dot{t}}{2m} \phi_1. \tag{3.17}$$

It is easy to see (using eq.(3.12)) that this generator reproduces the appropriate transformation (3.15)

$$\delta x^\mu(\tau) = \{x^\mu, G\} = \epsilon \frac{dx^\mu}{d\tau} \tag{3.18}$$

which is in agreement with Dirac's treatment [34][1]. Note that x^μ's are not gauge invariant variables in this case. This example shows that reparametrization symmetry can be identified with gauge symmetry.

We now fix the gauge symmetry by imposing a gauge condition. The standard choice is to identify the time coordinate t with the parameter τ

$$\phi_2 = t - \tau \approx 0 \; . \tag{3.19}$$

A straightforward computation of the algebra between the constraints (3.13, 3.19) (using eq.(3.12) once again) leads to the following second class set with:

$$\phi_{ab} = \{\phi_a, \phi_b\} = -2m\epsilon_{ab} \quad ; \quad (a, b = 1, 2) \tag{3.20}$$

where, ϵ_{ab} is an anti-symmetric tensor with $\epsilon_{12} = 1$.

The next step is to compute the DB(s) defined as

$$\{A, B\}_{DB} = \{A, B\} - \{A, \phi_a\}(\phi^{-1})_{ab}\{\phi_b, B\} \tag{3.21}$$

where A, B are any pair of phase-space variables and $(\phi^{-1})_{ab} = (2m)^{-1}\epsilon_{ab}$ is the inverse of ϕ_{ab}. It then follows

$$\{x, x\}_{DB} = \{p_x, p_x\}_{DB} = 0 \qquad ; \qquad \{x, p_x\}_{DB} = 1 \; . \tag{3.22}$$

The expected canonical bracket structure in the usual $2-d$ reduced phase-space comprising of variables x and p_x only is thus reproduced. The DB(s) imply a strong imposition of the second class constraints (ϕ_a). Consistent with this, $\{t, x\}_{DB} = 0$ showing that there is no spacetime noncommutativity if a gauge-fixing condition like eq.(3.19) is chosen. A question which now arises naturally is whether spacetime (or space-space) noncommutativity can be obtained by imposing a suitable variant of the gauge fixing condition (3.19). Before answering this question, we emphasize at this point that the DB(s) between various gauges should be related by suitable gauge transformations[2]. This idea will be useful in the sequel.

[1] In this treatment, the generator is a linear combination of the first class constraints. Since we have only one first class constraint ϕ_1 in the theory, the gauge generator is proportional to ϕ_1.

[2] We show (see appendix A) how this is done for a free Maxwell theory where the DB between phase-space variables in radiation and axial gauges are related by appropriate gauge transformations.

In the present case, the same procedure, as done (in the appendix) for a free Maxwell theory, is adopted to get hold of a set of variables $x^{'}$, $t^{'}$ satisfying a noncommutative algebra

$$\{t^{'}, x^{'}\}_{DB} = \theta \tag{3.23}$$

with θ being constant. The transformations (3.15) are written in terms of phase-space variables after strongly implementing the constraint (3.19). In component notation, we then have:

$$t^{'} = t + \epsilon \tag{3.24}$$

$$x^{'} = x + \epsilon\frac{dx}{d\tau} = x + \epsilon\frac{p_x}{m} \ . \tag{3.25}$$

Substitution of the above transformations in the L.H.S. of eq.(3.23) and using the Dirac algebra (3.22) for the unprimed variables, fixes ϵ to be:

$$\epsilon = -\theta p_x \ . \tag{3.26}$$

This shows the desired gauge fixing condition to be

$$t^{'} + \theta p_x - \tau \approx 0 \ . \tag{3.27}$$

Now one can just drop the prime to rewrite eq.(3.27) as

$$t + \theta p_x - \tau \approx 0 \ . \tag{3.28}$$

As one might expect, a direct calculation of the DB in this gauge immediately reproduces the noncommutative structure $\{t, x\}_{DB} = \theta$.

The analysis carried out above can be generalised trivially to higher $d+1$-dimensional Galilean spacetime. In the case of $d \geq 2$, one can see that the above spacetime noncommutativity is of the form $\{x^0, x^i\}_{DB} = \theta^{0i}$; $(x^0 = t)$. This can be derived by writing the counterpart of the transformations (3.24, 3.25) for $d \geq 2$ as:

$$x^{'0} = x^0 + \epsilon \tag{3.29}$$

$$x^{'i} = x^i + \epsilon\frac{dx^i}{d\tau} = x^i + \epsilon\frac{p^i}{m} \ . \tag{3.30}$$

Substituting back in the L.H.S. of $\{x'^0, x'^i\} = \theta^{0i}$, fixes ϵ to be:

$$\epsilon = -\theta^{0i}p_i \ . \tag{3.31}$$

The desired gauge fixing condition (dropping the prime) now becomes

$$x^0 + \theta^{0i}p_i - \tau \approx 0 \tag{3.32}$$

which is the analogue of (3.28). The space-space algebra for $d \geq 2$ is also noncommutative

$$\{x^i, x^j\}_{DB} = -\frac{1}{m}\left(\theta^{0i}p^j - \theta^{0j}p^i\right) . \tag{3.33}$$

The remaining non-vanishing DB(s) are

$$\{x^i, p_0\}_{DB} = -\frac{p^i}{m} \qquad \{x^i, p_j\}_{DB} = \delta^i{}_j \ . \tag{3.34}$$

The above forms of the DB(s) show a Lie-algebraic structure for the brackets involving phase-space variables (with the inclusion of identity). Following [33], an appropriate "diamond product" can be associated for this, in order to compose any pair of phase-space functions.

We have thus systematically derived the non-standard gauge condition leading to a noncommutative algebra. Also, the change of variables mapping this noncommutative algebra with the usual (commutative) algebra is found to be a gauge transformation.

There is yet another interesting way of deriving the Dirac algebra if one looks at the symplectic two-form $\omega = dp_\mu \wedge dx^\mu$ and then simply impose the conditions on p_0 and x^0, for all cases discussed. We consider the simplest case here. In $1+1$-dimension, the two-form ω can be written as

$$\omega = dp_t \wedge dt + dp_x \wedge dx \ . \tag{3.35}$$

Now imposing the condition on p_t (3.13) and t (3.19), we get:

$$\omega = -\frac{p_x}{m}dp_x \wedge d\tau + dp_x \wedge dx \ . \tag{3.36}$$

Note that the first term on the right hand side of the above equation vanishes as τ is not a variable in the configuration space. Now the inverse of the components of the two-form yields the brackets (3.22).

In the non-standard gauge (3.28), the two-form ω reads

$$\omega = dp_t \wedge dt - \frac{1}{\theta} dt \wedge dx \tag{3.37}$$

once the condition on p_x from eq.(3.28) is imposed. The inverse of the components of the two-form can be computed in a straightforward way to obtain the noncommutative structure $\{t, x\} = \theta$. The same procedure can be followed for the other cases discussed in the chapter. The role of integral curves within this symplectic formalism [35] is discussed in appendix B.

3.2 Free relativistic particle

In this section we take up the case of a free relativistic particle and study how spacetime noncommutativity can arise in this case also through a suitably modified gauge fixing condition. We start with the standard reparametrization invariant action of a relativistic free particle which propagates in $d+1$-dimensional "target spacetime"

$$S_0 = -m \int d\tau \sqrt{-\dot{x}^2} \tag{3.38}$$

with spacetime coordinates x^μ, $\mu = 0, 1, ...d$, the dot denoting differentiation with respect to the evolution parameter τ, and the Minkowski metric is $\eta = diag(-1, 1, ..., 1)$. In contrast to the NR case, the action here is already in the reparametrized form with all x^μ's (including $x^0 = t$) contained in the configuration space. The canonically conjugate momenta are given by

$$p_\mu = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}} \tag{3.39}$$

and satisfy the standard PB relations

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu \qquad ; \qquad \{x^\mu, x^\nu\} = \{p^\mu, p^\nu\} = 0. \tag{3.40}$$

Taking the square of eq.(3.39), it is easy to see that these are subject to the Einstein constraint

$$\phi_1 = p^2 + m^2 \approx 0 \ . \tag{3.41}$$

The reparametrization symmetry of the problem (under which the action (3.38) is invariant) can now be used together with the fact that $x^\mu(\tau)$ transforms as a scalar under world-line reparametrization (3.14), to find the infinitesimal transformation of the spacetime coordinate (3.15). As before, to derive the generator of the reparametrization invariance we write the variation in the Lagrangian as a total derivative:

$$\delta L = \frac{dB}{d\tau} \quad ; \quad B = -m\epsilon\sqrt{-\dot{x}^2} \ . \tag{3.42}$$

The generator of the infinitesimal transformation of the spacetime coordinate (3.18) can then be obtained from the usual Noether's prescription[3]

$$\begin{aligned} G &= \frac{1}{2}\left(p^\mu \delta x_\mu - B\right) \\ &= \frac{1}{2}\left(p^\mu \epsilon \frac{dX_\mu}{d\tau} + m\epsilon\sqrt{-\dot{x}^2}\right) \\ &= \frac{\epsilon\sqrt{-\dot{x}^2}}{2m}\phi_1 \end{aligned} \tag{3.43}$$

where we have used eq.(s) (3.15, 3.42).

A gauge condition can now be imposed to curtail the gauge freedom just as in the NR case. The standard choice is to identify the time coordinate x^0 with the parameter τ

$$\phi_2 = x^0 - \tau \approx 0 \tag{3.44}$$

which is the analogue of eq.(3.19). The constraints (3.41, 3.44) form a second class set with

$$\{\phi_a, \phi_b\} = 2p_0\epsilon_{ab} \ . \tag{3.45}$$

The resulting non-vanishing DB(s) are

$$\{x^i, p_0\}_{DB} = \frac{p^i}{p_0} \quad ; \quad \{x^i, p_j\}_{DB} = \delta^i{}_j \tag{3.46}$$

which imposes the constraints ϕ_1 and ϕ_2 strongly. In particular, we observe $\{x^0, x^i\}_{DB} = 0$, showing that there is no spacetime noncommutativity. This is again consistent with the fact that the constraint (3.44) is now strongly imposed. Taking a cue from our previous NR example,

[3]The factor of 1/2 comes from symmetrization. To make this point clear, we must note that while computing $\{x^\mu, G\}$, an additional factor of 2 crops up from the bracket between x^μ and δx_μ as δx_μ is related to p_μ by the relations (3.15) and (3.39). The factor of 1/2 is placed in order to cancel this additional factor of 2.

we see that we must have a variant of eq.(3.44) as a gauge fixing condition to get spacetime noncommutativity in the following form

$$\{x^{'0}, x^{'i}\}_{DB} = \theta^{0i} \tag{3.47}$$

(θ^{0i} being constants) where $x^{'\mu}$ denotes the appropriate gauge transforms of x^{μ} variables. The transformed variables $x^{'\mu}$ in terms of the variables x^{μ} can be determined by considering an infinitesimal transformation (3.15) written in terms of phase-space variables as

$$x^{'0} = x^0 + \epsilon \qquad ; \qquad x^{'i} \;=\; x^i - \epsilon \frac{p^i}{p_0} \tag{3.48}$$

where we have used the relation $\frac{dx^i}{d\tau} = -\frac{p^i}{p_0}$ obtained from eq.(3.39). A simple inspection after substituting the above relations (3.48) back in eq.(3.47) and using eq.(3.46), shows that ϵ is given by

$$\epsilon = -\theta^{0i} p_i \tag{3.49}$$

which is identical to eq.(3.31). Hence the gauge transformed variables $x^{'\mu}$ (3.48) for the above choice of ϵ are given by:

$$x^{'0} = x^0 - \theta^{0i} p_i \tag{3.50}$$

$$x^{'i} \;=\; x^i + \theta^{0j} p_j \frac{p^i}{p_0} \;. \tag{3.51}$$

The above set of transformations and the relation (3.46), leads to the following Dirac algebra between the primed variables

$$\{x^{'0}, x^{'i}\}_{DB} = \theta^{0i} \tag{3.52}$$

$$\{x^{'i}, x^{'j}\}_{DB} = \frac{1}{p_0}\left(\theta^{0i} p^j - \theta^{0j} p^i\right) \tag{3.53}$$

$$\{x^{'i}, p_0^{'}\}_{DB} = \frac{p^i}{p_0} \qquad ; \qquad \{x^{'i}, p_j^{'}\}_{DB} = \delta^i{}_j \;. \tag{3.54}$$

Note that unlike x's, p's are gauge invariant objects as $\{p^\mu, \phi\} = 0$; hence $p'_\mu = p_\mu$.

It is interesting to observe that the solution of the gauge parameter ϵ remains the same in both the relativistic case as well as the NR case. Also, m in the NR case gets replaced by $-p_0$ in the relativistic case. With this identification, one can easily see that the complete Dirac algebra in the NR case goes over to the corresponding algebra in the relativistic case. However, since p_0 does not have a vanishing bracket with all other phase-space variables, its occurence in the denominators in eq.(s) (3.53, 3.54) shows that the bracket structure of the phase-space variables in the relativistic case is no longer Lie-algebraic, unlike the NR case discussed in the previous section.

Furthermore, the modified gauge fixing condition is given by:

$$\phi_2 = x^0 + \theta^{0i} p_i - \tau \approx 0 \qquad , \qquad i = 1, 2, ...d \ . \tag{3.55}$$

It is trivial to check that the constraints (3.41, 3.55) also form a second class pair as

$$\{\phi_a, \phi_b\} = 2p_0 \epsilon_{ab} \ . \tag{3.56}$$

The set of non-vanishing DB(s) consistent with the strong imposition of the constraints (3.41, 3.55) reproduces the results (3.52, 3.53, 3.54). Eq.(3.54) is the same as in the standard gauge (3.44), while eq.(3.53) implies non-trivial bracket relations among spatial coordinates upon imposition of the gauge fixing condition (3.55).

It should be noted that the above gauge fixing condition (3.55) was also given in [32]. Indeed a change of variables, which is different from eq.(s) (3.50, 3.51), is found there by inspection, using which the spacetime noncommutativity gets removed. However, the change of variables that we find here is related to a gauge transformation providing in turn a systematic derivation of the modified gauge condition and also spacetime noncommutativity. Moreover, the definition of the Lorentz generators (rotations and boosts) in ([32]) requires some additional terms (in the modified gauge) in order to have a closed algebra between the generators. In our approach, the definition of the Lorentz generators remains unchanged, simply because these are gauge invariant.

The Lorentz generators (rotations and boosts) are defined as:

$$M_{ij} = x_i p_j - x_j p_i \tag{3.57}$$

$$M_{0i} = x_0 p_i - x_i p_0. \tag{3.58}$$

As expected, they satisfy the usual algebra in both the unprimed and the primed coordinates as $M_{\mu\nu}$ and p_μ are both gauge invariant.

$$\{M_{ij}, p_k\}_{DB} = \delta_{ik} p_j - \delta_{jk} p_i \tag{3.59}$$

$$\{M_{ij}, M_{kl}\}_{DB} = \delta_{ik} M_{jl} - \delta_{jk} M_{il} + \delta_{jl} M_{ik} - \delta_{il} M_{jk} \tag{3.60}$$

$$\{M_{ij}, M_{0k}\}_{DB} = \delta_{ik} M_{0j} - \delta_{jk} M_{0i} \tag{3.61}$$

$$\{M_{0i}, M_{0j}\}_{DB} = M_{ji}. \tag{3.62}$$

However, the algebra between the space coordinates and the rotations, boosts are different in the two gauges (3.44, 3.55). This is not surprising as x^k is not gauge invariant under gauge transformation. We find

$$\{M_{ij}, x^k\}_{DB} = \delta_i{}^k x_j - \delta_j{}^k x_i \tag{3.63}$$

$$\{M_{0i}, x^j\}_{DB} = x_i \frac{p^j}{p_0} - x_0 \delta_i{}^j \tag{3.64}$$

$$\begin{aligned} \{M_{ij}, x'^k\}_{DB} &= \{M_{ij}, x^k + \theta^{0l} p_l \frac{p^k}{p_0}\}_{DB} \\ &= \delta_i{}^k x'_j - \delta_j{}^k x'_i + \frac{1}{p_0}\left(\theta^0{}_i p^k p_j - \theta^0{}_j p^k p_i\right) \end{aligned} \tag{3.65}$$

$$\begin{aligned} \{M_{0i}, x'^j\}_{DB} &= \{M_{0i}, x^j + \theta^{0l} p_l \frac{p^j}{p_0}\}_{DB} \\ &= x'_i \frac{p^j}{p_0} - x'_0 \delta_i{}^j - \theta^0{}_i p^j \end{aligned} \tag{3.66}$$

where we have used eq.(3.51) and the algebra (3.46). The same results can also be obtained using the relations (3.52, 3.53, 3.54).

Now we note that the gauge choice (3.55) is not Lorentz invariant. Yet the Dirac bracket procedure forces this constraint equation to be strongly valid in all Lorentz frames [36]. This can be made consistent if and only if an infinitesimal Lorentz boost to a new frame[4]

$$p^{\mu} \to p^{'\mu} = p^{\mu} + \omega^{\mu\nu} p_{\nu} \tag{3.67}$$

is accompanied by a compensating infinitesimal gauge transformation

$$\tau \to \tau^{'} = \tau + \Delta\tau. \tag{3.68}$$

The change in x^{μ}, upto first order, is therefore

$$\begin{aligned} x^{'\mu}(\tau) &= x^{\mu}(\tau^{'}) + \omega^{\mu\nu} x_{\nu}(\tau) \\ &= x^{\mu}(\tau) + \Delta\tau \frac{dx^{\mu}}{d\tau} + \omega^{\mu\nu} x_{\nu}. \end{aligned} \tag{3.69}$$

In particular, the zeroth component is given by:

$$x^{'0}(\tau) = x^{0}(\tau) + \Delta\tau \frac{dx^{0}}{d\tau} + \omega^{0i} x_{i}. \tag{3.70}$$

Since the gauge condition (3.55) is $x^{0}(\tau) \approx \tau - \theta^{0i} p_{i}$, $x^{'0}(\tau)$ also must satisfy $x^{'0}(\tau) = (\tau - \theta^{0i} p_{i}^{'})$ in the boosted frame, which can now be written using eq.(3.67), as

$$\begin{aligned} x^{'0}(\tau) &= \tau - \theta^{0i} p_{i}^{'} \\ &= \tau - \theta^{0i} p_{i} + \theta^{0i} \omega^{0i} p_{0}. \end{aligned} \tag{3.71}$$

Comparing with eq.(3.70) and using the gauge condition (3.55), we can now solve for $\Delta\tau$ to get:

$$\Delta\tau = \frac{\theta^{0i}\omega^{0i} p_{0} - \omega^{0i} x_{i}}{1 - \theta^{0i} \dot{p}_{i}} \quad ; \quad \dot{p}_{i} = \frac{dp_{i}}{d\tau} \ . \tag{3.72}$$

[4]A similar treatment as in [36] has been given in [37] for a free relativistic particle coupled to Chern-Simons term.

The spatial components of eq.(3.69) (for a pure boost) therefore satisfy

$$\begin{aligned}
\delta x^j(\tau) &= x'^j(\tau) - x^j(\tau) = \Delta\tau \frac{dx^j}{d\tau} + \omega^{j0} x_0 \\
&= \omega^{0i} \left(x_i \frac{p^j}{p_0} - x_0 \delta_i{}^j - \theta^{0i} p^j \right) .
\end{aligned} \tag{3.73}$$

Hence we find that eq.(3.73) and eq.(3.66) are consistent with each other. However, note that in the above derivation we have taken θ^{0i} to be a constant. If we take θ^{0i} to transform as a tensor, then for a Lorentz boost to a new frame, it changes as

$$\theta^{0i} \rightarrow \theta'^{0i} = \theta^{0i} + \omega^{0j} \theta^{ji} \tag{3.74}$$

and the entire consistency program would fail. The $(1+1)$–dimensional case is special, since even if we take θ^{01} to transform as a tensor, this will not affect the consistency program as it remains invariant ($\theta'^{01} = \theta^{01}$) under Lorentz boost.

Now in [17] we have shown that there exists some special values of the reparametrization parameter ϵ which leads to noncommuting structures falling in the Lie-algebraic category [33]. Setting

$$\epsilon = -\theta^{0k} p_k \frac{p_0}{m} \tag{3.75}$$

and using eq.(3.46) and eq.(3.48), we obtain the following algebra between the primed coordinates:

$$\{x'^i, x'^j\}_{DB} = \frac{1}{m} \left(\theta^{0i} p^j - \theta^{0j} p^i \right) \tag{3.76}$$

$$\{x'^0, x'^i\}_{DB} = \frac{1}{m} (\theta^{0i} p_0 + \theta^{0k} p_k \frac{p^i}{p_0}) \tag{3.77}$$

$$\{x'^i, p'_0\}_{DB} = \frac{p^i}{p_0} \qquad ; \qquad \{x'^i, p'_j\}_{DB} = \delta^i{}_j \ . \tag{3.78}$$

It is now important to observe that the noncommutativity in the space-space coordinates (3.76) has a Lie-algebraic structure in phase-space (with the inclusion of identity) and not in spacetime.

This is in contrast to the results derived for the relativistic free particle where space-space noncommutativity (eq.(3.53)) was not Lie-algebraic.

The above solution of ϵ (3.75) shows that the desired gauge fixing condition is given by:

$$\phi_3 = x^0 + \theta^{0k} p_k \frac{p_0}{m} - \tau \approx 0, \qquad k = 1, 2, ...d. \tag{3.79}$$

It is easy to check that the constraints (3.41, 3.79) once again form a second class pair (3.56). The set of non-vanishing DB(s) consistent with the strong imposition of the constraints (3.41, 3.79) reproduces the results (3.76, 3.77, 3.78).

Another interesting choice of ϵ is the following:

$$\epsilon = -d_k \theta^{kl} p_l \frac{p_0}{m} \tag{3.80}$$

where, d_k are arbitrary dimensionless constants.

This yields (using eq.(3.46) and eq.(3.48)) the following algebra between the primed coordinates:

$$\{x'^i, x'^j\}_{DB} = \frac{d_k}{m} \left(\theta^{ki} p^j - \theta^{kj} p^i \right) \tag{3.81}$$

$$\{x'^0, x'^i\}_{DB} = \frac{d_k}{m} \left(\theta^{ki} p_0 + \theta^{kl} p_l \frac{p_i}{p_0} \right) \tag{3.82}$$

$$\{x'^i, p_0'\}_{DB} = \frac{p_i}{p_0} \qquad ; \qquad \{x'^i, p_j'\}_{DB} = \delta^i{}_j \ . \tag{3.83}$$

Once again we obtain a Lie-algebraic noncommutative structure in the space-space sector. However, note that eq.(3.81) is different from eq.(3.76) because the noncommutative parameter θ in eq.(3.81) has space indices in contrast to the spacetime indices appearing in eq.(3.76). The spacetime algebra is once again not Lie-algebraic in form.

The desired gauge fixing condition which lead to the above DB(s) read:

$$\phi_4 = x^0 + d_k \theta^{kl} p_l \frac{p_0}{m} - \tau \approx 0, \qquad k = 1, 2, ...d. \tag{3.84}$$

The algebra of the Lorentz generators for the above choices of the reparametrization parameter ϵ can be investigated in a similar way as for the relativistic free particle and once again the internal consistency of our analysis can be established.

Finally, there exists choices of ϵ for which the space-space algebra can be made to vanish. The choices are:

$$\epsilon = e_k \theta^{0k} \frac{p_0^2}{m} \tag{3.85}$$

and

$$\epsilon = -f_{kl} \theta^{kl} p_0 \tag{3.86}$$

where, e_k and f_{kl} are arbitrary dimensionless constants.

The spacetime algebras however do not vanish for the above values of ϵ and are as follows:

$$\{x'^0, x'^i\} = \frac{2e_k}{m} \theta^{0k} p^i \tag{3.87}$$

and

$$\{x'^0, x'^i\} = f_{kl} \theta^{kl} \frac{p^i}{p_0} \ . \tag{3.88}$$

Let us now make certain observations. Although, the relations (3.33), (3.53), (3.76) and (3.81) are reminescent of Snyder's algebra [4], there is a subtle difference. This can be seen by noting that the right hand side of these relations do not have the structure of an angular momentum operator in their differential representation (obtained by repacing p_j by $(-i\partial_j)$) in contrast to Snyder's algebra. Further, the relations (3.76) and (3.81) has a similar structure to the commutation relations describing the Lie-algebraic deformation of the Minkowski space [38], the only difference being that momentum operators appear at the right hand side of the relations instead of the position operators.

Now in the cases where the noncommutativity takes the canonical structure ($[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$), one can infer the presence of non-locality from the fact that two localised functions f and g having supports within a size $\delta << \sqrt{||\theta||}$, yields a function $f \star g$ which is non-vanishing over a much larger region of size $||\theta||/\delta$ [23]. One therefore expects a similar qualitative feature of non-locality arising from the "diamond product" appropriate for the Lie-bracket structure of noncommutativity in the NR case also. This is further reinforced by the fact that coordinate transformations (3.29, 3.30) involve mixing of coordinates and momenta. Since this mixing is present in the relativistic case as well (eq.(s) (3.50, 3.51)), it is expected to maintain the non-locality of the noncommutative theory, although an appropriate "diamond product" cannot be readily constructed because of the absence of a Lie-bracket structure. Also, the mixing of coordinates and momenta is a natural consequence of our gauge conditions which essentially involve phase-space variables interpolating between the commutative and noncommutative descriptions.

Besides, spacetime noncommutativity arising from a relation like eq.(3.52), implies that the "co-ordinate" time $\hat{x}^0$ cannot be localised as any state will have a spread in the spectrum of $\hat{x}^0$. This eventually leads to the failure of causality and violation of locality in quantum field theory [39, 40].

3.3 Interaction with background Electromagnetic Field

In this section, we consider interactions with a background electromagnetic field which still keeps the time reparametrization symmetry of the relativistic free particle intact. Before going over to the general case, we consider a constant background field. The interaction term to be added to S_0 is then

$$S_F = -\frac{1}{2}\int d\tau F_{\mu\nu}x^{\mu}\dot{x}^{\nu} \tag{3.89}$$

where, $F_{\mu\nu}$ is a constant field strength tensor. The canonical momenta are given by

$$\Pi_{\mu} = p_{\mu} + \frac{1}{2}F_{\mu\nu}x^{\nu} \tag{3.90}$$

where, p_{μ} is given by eq.(3.39). The Einstein constraint (3.41) which is the first class constraint of the theory once again follows from the reparametrization symmetry of the model. The PB(s) are[5]

$$\{x^{\mu}, p_{\nu}\} = \delta^{\mu}_{\nu} \quad ; \quad \{x^{\mu}, x^{\nu}\} = 0 \quad ; \quad \{p_{\mu}, p_{\nu}\} = -F_{\mu\nu}. \tag{3.91}$$

Note that p_{μ} does not have zero PB with the constraint (3.41) anymore and thus is not gauge invariant. Now to obtain the generator of reparametrization symmetry, we again exploit the

[5]These relations follow from the basic canonical algebra $\{x_{\mu}, \Pi^{\nu}\} = \delta^{\nu}_{\mu}$; $\{x_{\mu}, x_{\nu}\} = \{\Pi_{\mu}, \Pi_{\nu}\} = 0$.

infinitesimal transformation of the spacetime coordinate given by eq.(3.15). Proceeding exactly as in the earlier sections, we write the variation of the Lagrangian in a total derivative form as:

$$\delta L = \frac{dB}{d\tau} \quad ; \quad B = -m\epsilon\sqrt{-\dot{x}^2} - \frac{\epsilon}{2}F_{\mu\nu}x^{\mu}\frac{dx^{\nu}}{d\tau} \ . \tag{3.92}$$

Then the generator is obtained from usual Noether's prescription (as it was done for the case of the free relativistic particle), by making use of eq.(3.90) to get

$$\begin{aligned} G &= \frac{1}{2}\left(\Pi^{\mu}\delta x_{\mu} - B\right) \\ &= \frac{\epsilon\sqrt{-\dot{x}^2}}{2m}\left[\Pi^{\mu}p_{\mu} + m^2 + \frac{1}{2}F_{\mu\nu}x^{\mu}p^{\nu}\right] \\ &= \frac{\epsilon\sqrt{-\dot{x}^2}}{2m}\phi_1 \end{aligned} \tag{3.93}$$

where, $\phi_1 = p^2 + m^2 \approx 0$ is the first class constraint (3.41). This clearly generates the infinitesimal transformation of the spacetime coordinate (3.18). Hence we have again shown that the generator is indeed proportional to the first class constraint which is in conformity with Dirac's treatment. Further, the relation between reparametrization symmetry and gauge symmetry becomes evident once more. Now the gauge/reparametrization symmetry can be fixed by imposing a gauge condition. The standard choice is given by eq.(3.44). The constraints (3.41, 3.44) form a second class set with the PB(s) between them given by eq.(3.45). So the non-vanishing DB(s) are given by eq.(3.46) and

$$\{p_i, p_j\}_{DB} = -F_{ij} \quad ; \quad \{p_0, p_i\}_{DB} = F_{ij}\frac{p_j}{p_0} \ . \tag{3.94}$$

To obtain noncommutativity between the primed set of spacetime coordinates (3.47), we first observe that the zeroth component and spatial components of eq.(3.15) (in the standard gauge (3.44)) leads to eq.(3.48) where we have used the relation $\frac{dx^i}{d\tau} = -\frac{p_i}{p_0}$ obtained from eq.(3.39). Using the relations (3.47, 3.48) fixes the value of ϵ, which, in view of the non-vanishing bracket (3.94), turns out to be

$$\epsilon = -\theta^{0j}P_j \tag{3.95}$$

where,

$$P_{\mu} = p_{\mu} + F_{\mu\nu}x^{\nu} \tag{3.96}$$

is gauge invariant since $\{P_\mu, p_\nu\} = 0$. As a simple consistency check, we observe that the solution (3.95) reduces to the free particle solution (3.49) for vanishing electromagnetic field. It should also be noted that the non-vanishing DB(s) involving P_μ in the standard gauge (3.44) are given by:

$$\{x^i, P_j\}_{DB} = \delta^i{}_j \quad ; \quad \{P_\mu, P_\nu\}_{DB} = F_{\mu\nu} \quad ; \quad \{x^i, P_0\}_{DB} = \frac{p^i}{p_0} \ . \tag{3.97}$$

The set of transformations relating the unprimed and primed coordinates can now be written down using eq.(s) (3.48) and (3.95):

$$x'^0 = x^0 - \theta^{0i} P_i \tag{3.98}$$

$$x'^i = x^i - \theta^{0j} P_j \frac{dx^i}{d\tau} = x^i + \theta^{0j} P_j \frac{p^i}{p_0} \tag{3.99}$$

where we have used the relation $\frac{p'_j}{p'_0} = -\frac{dx'_j}{d\tau}$ since $\frac{dx^0}{d\tau} = 1$ in the old gauge (3.44). From the above set of transformations and the relations (3.46, 3.94, 3.97), we compute the DB(s) between the primed variables:

$$\{x'^0, x'^i\}_{DB} \;\; = \;\; \theta^{0i} \tag{3.100}$$

$$\begin{aligned}\{x'^i, x'^j\}_{DB} &= \frac{1}{p_0}\left(\theta^{0i} p^j - \theta^{0j} p^i\right) \\ &= \frac{1}{p'_0}\left(\theta^{0i} p'^j - \theta^{0j} p'^i\right) + O(\theta^2).\end{aligned} \tag{3.101}$$

In order to express the variables on the R.H.S. in terms of primed ones[6], use has been made of eq.(3.99) to get:

$$\frac{p'_j}{p'_0} = \frac{p_j}{p_0} - \theta^{0k} P_k \frac{d}{d\tau}\left(\frac{p_j}{p_0}\right) + O(\theta^2). \tag{3.102}$$

Observe that the change of variables (3.98, 3.99) leading to the algebra among the primed variables, are basically infinitesimal gauge transformations that are valid to first order in the reparametrization parameter ϵ. Moreover, from eq.(3.95) it follows that ϵ is proportional to θ.

[6] Note that, since P_μ (eq.(3.96)) is gauge invariant, $P'_\mu = P_\mu$.

Hence, the Dirac algebra (3.100, 3.101) between the primed variables are also valid upto order θ. However, it turns out that these results are actually exact, as we shall now show below.

As before, it is possible to write down the modified gauge condition from the solution (3.95) for ϵ as

$$\phi_2 = x^0 + \theta^{0i} P_i - \tau \approx 0, \qquad i = 1, 2, ...d. \tag{3.103}$$

The constraints (3.41, 3.103) again form a second class set with the PB(s) between them being given by (3.45). So we recover the previous DB(s) (3.100, 3.101) between spacetime coordinates x^μ.

Finally we consider the coupling of the relativistic free particle to an arbitrary electromagnetic field. As before the action is reparametrization invariant. Here we replace eq.(3.89) by

$$S_F = -\int d\tau A_\mu(x)\dot{x^\mu}. \tag{3.104}$$

The choice $A_\mu = -\frac{1}{2} F_{\mu\nu} x^\nu$ for constant $F_{\mu\nu}$ reproduces the action (3.89). The Einstein constraint (3.41) and PB(s) (3.91) again follow. The canonical momenta are given by:

$$\Pi_\mu = p_\mu - A_\mu \tag{3.105}$$

where, p_μ is defined by eq.(3.39). The gauge symmetry can be fixed by imposing a gauge condition. The standard choice is given by eq.(3.44). The constraints (3.41, 3.44) form a second class set with the PB(s) between them again given by eq.(3.45). So the non-vanishing DB(s) are given by eq.(s) (3.46) and (3.94). As before, exploiting the reparametrization symmetry of the problem, the infinitesimal transformation of the spacetime coordinate is given by eq.(3.15) which leads to eq.(3.48) in the standard gauge (3.44) (where we have again used the relation $\frac{dx^i}{d\tau} = -\frac{p^i}{p_0}$ obtained from eq.(3.39)). Demanding noncommutativity between the primed set of spacetime coordinates by imposing the condition (3.47) and using the relation (3.48) leads to:

$$\{x^0 + \epsilon, x^i - \epsilon \frac{p^i}{p_0}\}_{DB} = \theta^{0i} \tag{3.106}$$

which fixes the value of ϵ to be

$$\epsilon = -\theta^{0j} p_j + O(\theta^2). \tag{3.107}$$

Here we are content with expression linear in θ as a gauge invariant P_μ (counterpart of eq.(3.96)) cannot be defined here.

A gauge condition (which is the same as eq.(3.55)) can be identified once again leading to noncommutativity between spacetime coordinates. The computation of the DB between the spacetime coordinates in this gauge gives:

$$\{x^0, x^i\}_{DB} = \frac{\theta^{0i}}{1 + \theta^{0j} F_{j\mu} \frac{p^\mu}{p_0}} \tag{3.108}$$

which has already been given in [32]. One can easily see that to the linear order in θ, the above result goes to eq.(3.47).

3.4 Summary

We have discussed an approach whereby both space-space as well as spacetime noncommutative stuctures are obtained in a particular (non-standard gauge) in models having reparametrization invariance. These structures are obtained by calculating either DB(s) or symplectic brackets and the results agree. We have also shown that the noncommutative results in the non-standard gauge and the commutative results in the standard gauge are gauge transforms of each other. In other words, equivalent physics is described by working either with the usual brackets or the noncommuting brackets. We feel our approach is conceptually cleaner and more elegant than those [32] where such change of variables are found by inspection leading to ambiguities in the definition of physical (gauge invariant) variables and apparently lacking any connection with the symmetries of the problem. For instance, the angular momentum operator gets modified in distinct gauges, by appropriate inclusion of extra terms, so that the closure property is satisfied. In our approach, on the contrary, the angular momentum remains invariant since the change of variables is just a gauge transformation. Consequently we do not find these extra terms appearing. We also feel that the present approach could be useful in illuminating the role of variable changes used for relating the commuting and noncommuting descriptions in field theory.

3.5 Appendix A

Here we would like to show how the DB(s) for any pair of variables, computed for Coulomb and axial gauges, are connected through gauge transformations. For that we consider the action of free Maxwell theory

$$S = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu}. \tag{3.109}$$

The first class constraints of the theory responsible for generating gauge transformations are

$$\pi_0(x) \approx 0 \quad ; \quad \partial_i \pi_i(x) \approx 0 \ . \tag{3.110}$$

The above set of constraints can be rendered second class by gauge fixing. We first consider the Coulomb gauge given by:

$$A_0 \approx 0 \qquad ; \qquad \partial_i A_i(x) \approx 0. \tag{3.111}$$

The DB computed between A_i, Π_j in this gauge yields the familiar transverse delta function [34], [36]:

$$\begin{aligned} \{A_i(x), \Pi_j(y)\}^{(c)}_{DB} &= -\left(\delta_{ij} - \frac{\partial_i \partial_j}{\partial^2}\right)\delta(x-y) \\ &= -\delta^T_{ij}\delta(x-y) \end{aligned} \tag{3.112}$$

where the superscript c denotes the Coulomb gauge.

The corresponding DB in axial gauge $A_3 \approx 0$ and $(\Pi_3 - \partial_3 A_0) \approx 0$ [7]is given by [34], [36]:

$$\{A_i(x), \Pi_j(y)\}^{(a)}_{DB} = -\delta_{ij}\delta(x-y) + \delta_{3j}\frac{\partial_i}{\partial_3}\delta(x-y). \tag{3.113}$$

Now the gauge field configurations $A_i^{(a)}$ and $A_i^{(c)}$ are connected by the gauge transformation

$$A_i^{(a)} = A_i^{(c)} + \partial_i \Lambda \tag{3.114}$$

[7]This follows by demanding time conservation of the gauge; i.e., $\partial_0 A_3 = \partial_0 A_3 - \partial_3 A_0 + \partial_3 A_0 = -\Pi_3 + \partial_3 A_0 \approx 0$.

where Λ is the gauge transformation parameter. Imposing $A_3^{(a)} = 0$ (axial gauge), fixes the value of Λ to be

$$\Lambda = -\frac{1}{\partial_3} A_3^{(c)} \tag{3.115}$$

so that

$$A_i^{(a)} = A_i^{(c)} - \frac{\partial_i}{\partial_3} A_3^{(c)}. \tag{3.116}$$

On the other hand, Π_i is gauge invariant, $\Pi_i^{(a)} = \Pi_i^{(c)}$. Hence, we have:

$$\{A_i(x), \Pi_j(y)\}_{DB}^{(a)} = \{A_i(x) - \frac{\partial_i}{\partial_3} A_3(x), \Pi_j(y)\}_{DB}^{(c)} \tag{3.117}$$

Using the Coulomb gauge result (3.112), the axial gauge algebra (3.113) is correctly reproduced.

3.6 Appendix B

We develop the symplectic formalism in this appendix and show the connection between integral curves and the Hamilton's equations of motion in the time-reparametrized version.

Let $Q = R \times Q_0$, $(Q_0 = q^i(t), i = 1, 2, ..., n)$, be a $n+1$–dimensional configuration space which includes time t. The corresponding phase-space Γ is $2n+2$–dimensional with coordinates (t, q^i, p_t, p_i). A function $F(t, q^i, p_t, p_i)$ on this phase-space is defined as follows:

$$F(t, q^i, p_t, p_i) = p_t + H_0(q^i, p_i). \tag{3.118}$$

Also let $\tilde{\theta} = p_t dt + p_i dq^i$ be a 1-form on Γ. Now let Σ be a sub-manifold of Γ defined by $F(t, q^i, p_t, p_i) = 0$. Restricting $\tilde{\theta}$ to Σ, we get:

$$\tilde{\theta}|_\Sigma = -H_0(q^i, p_i)dt + p_i dq^i. \tag{3.119}$$

An arbitrary tangent vector $\vec{X}$ to a curve in Σ is given by:

$$\vec{X} = u\frac{\partial}{\partial t} + v^j(q^i, p_i)\frac{\partial}{\partial q^j} + f_j(q^i, p_i)\frac{\partial}{\partial p_j} \tag{3.120}$$

with u, v^j and f_j's being arbitrary coefficients.

Demanding that the 2-form $\tilde{\omega} = d\tilde{\theta}|_\Sigma$ is degenerate, i.e., $\exists\ \vec{X} \neq 0$, such that upon contraction, the one-form $\tilde{\omega}(\vec{X}) = 0$, we immediately obtain the following equations:

$$f_i + u\frac{\partial H_0}{\partial q^i} = 0 \tag{3.121}$$

$$-v_i + u\frac{\partial H_0}{\partial p_i} = 0\ . \tag{3.122}$$

Hence eq.(3.120) can be written as

$$\vec{X} = u\left(\frac{\partial}{\partial t} + \frac{\partial H_0}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H_0}{\partial q^i}\frac{\partial}{\partial p_i}\right). \tag{3.123}$$

Now recall that an integral curve of a vector field is a curve such that the tangent at any point to this curve gives the value of the vector field at that point.

In general, any tangent vector field $\vec{X}$ to a family of curves, parametrised by τ, in the space Σ can be written as

$$\begin{aligned} \vec{X} &= \dot{x}^\mu \partial_\mu \quad ; \quad \dot{x}^\mu = \frac{dx^\mu}{d\tau} \\ &= \dot{t}\frac{\partial}{\partial t} + \dot{q}^i\frac{\partial}{\partial q^i} + \dot{p}_i\frac{\partial}{\partial p_i}\ . \end{aligned} \tag{3.124}$$

The equations of the integral curves (obtained by comparing eq.(s) (3.123, 3.124)) are given by:

$$\dot{q}^i = u\frac{\partial H_0}{\partial p_i} \quad , \quad \dot{t} = u \quad , \quad \dot{p}_i = -u\frac{\partial H_0}{\partial q^i}\ . \tag{3.125}$$

Note that we recover the usual Hamiltonian equations of motion in the $t = \tau$ gauge. It is the parameter u which is responsible for inducing the time reparametrization invariance.

Now we consider the example of a NR particle in $1+1$-dimension, the Hamiltonian of which reads:

$$H_0 = \frac{p_x^2}{2m}\ . \tag{3.126}$$

In $1+1$-dimension, the equations of the integral curves (3.125) can be rewritten as

$$\dot{x} = u\frac{\partial H_0}{\partial p_x} \quad , \quad \dot{t} = u \quad , \quad \dot{p_x} = -u\frac{\partial H_0}{\partial x}\ . \tag{3.127}$$

Substituting the form of the Hamiltonian (3.126) in eq.(3.127), we obtain:

$$p_x = \frac{m\dot{x}}{\dot{t}} = m\frac{dx}{dt} = constant \tag{3.128}$$

which is the equation of the integral curve. Note that the above form of the canonical momentum is independent of the parameter u. This establishes a connection between the integral curve on Σ and the canonical momenta. Also from eq.(s) (3.118, 3.126), we have:

$$p_t + \frac{p_x^2}{2m} = 0 \tag{3.129}$$

which is nothing but the first class constraint (3.13) in the time reparametrized version of the NR particle. Hence, the constraint of the time reparametrized theory is also obtained from the integral curve. The connection between the integral curves and the constraints for the other models discussed in the chapter can be shown in a similar way following the above approach.

Chapter 4

Seiberg-Witten map and violation of Galilean symmetry in a noncommutative planar system

As we have mentioned earlier, motivated by string theory, noncommutative spacetimes have drawn considerable attention in field theories [14], [23], [41], [42], quantum mechanics [43, 44, 45, 46, 47, 48, 49, 50, 51, 15] as well as for their phenomenological implications [52], [53], [54], [55], [56], [46], [47]. One of the most interesting things in noncommutative field theories is that even the $U(1)_\star$ gauge group has non-Abelian-like characterestics such as self-interactions.

On the other hand, investigations towards violation of Lorentz symmetry in noncommutative systems steming from a fundamental length scale provided by noncommutative parameter θ have gained considerable momentum in recent literature. It is generally assumed that there is no spacetime noncommutativity ($\theta^{0i} = 0$), in order to avoid any non-unitarity in quantum field theory based on it. Another reason for assuming $\theta^{0i} = 0$ is to avoid higher order time derivatives in the action[1]. Clearly, the condition $\theta^{0i} = 0$ spoils the manifest Lorentz symmetry right in the beginning and this is true, irrespective of whether one works with the original

[1] In a series of fundamental papers Doplicher et.al [57, 58] have however shown in complete generality that one can construct unitary quantum field theory even when $\theta^{0i} \neq 0$.

theory involving noncommutative variables or with an equivalent effective theory in terms of ordinary commutative variables obtained by SW map [14], [59], whenever applicable[2]. These two methods of analysis need not always be equivalent. For example, the IR problem found in noncommutative field theory [60, 61] is not present in the commutative variable approach [62], revealing an equivalence at best on a perturbative level. For the latter method, one can, for example, consider the action of $U(1)_\star$ Maxwell gauge theory, which when rewritten in terms of commutative variables using SW map, develops certain θ-dependent terms, in addition to the standard ones, which are manifestly Lorentz invariant (non-invariant) if $\theta^{\mu\nu}$ transforms like a tensor (non-tensor and fixed for all frames) [63]. These two cases correspond to observer and particle Lorentz transforms [64]. It is to be noted that the violation of Lorentz symmetry by extremely tiny $\theta^{\mu\nu}$ term is relevant at a very short distance or equivalently at a very high energy scale. Consequently, these additional θ-dependent correction terms can be treated as perturbations. As a result, the noncommutative quantum field theory is practically considered Lorentz invariant in zeroth order in $\theta^{\mu\nu}$, with the first order corrections coming from the expansion of star product and SW map. Various aspects of noncommutative quantum mechanics have also been studied, which are usually formulated through Schrödinger equation written in terms of noncommutative wave functions $\hat{\psi}$. Clearly this is in the NR framework. The presence of the star product can give rise to some new features like, say in presence of potential terms, the star product expansion gives rise to a Bopp shift [65], [66], [43] in the arguement of the potential. Besides, the presence of exotic Galilean symmetry have also been found in various noncommutative quantum mechanical model [30].

In this chapter, we study a planar noncommutative NR system coupled to a $U(1)_\star$ gauge field. Since the above mentioned condition $\theta^{0i} = 0$ is Galilean invariant, it is therefore quite interesting to look for any violation in Galilean symmetry in any NR noncommutative system where matter field is coupled to noncommutative gauge fields. As in their relativistic counterparts (as mentioned above), we shall be looking for this violation through an effective theory obtained

[2]By the term "applicable", we mean that SW map can be only applied to fields which transform appropriately under gauge transformation in presence of a gauge symmetry. For example, SW map cannot be applied to real scalar fields, which do not transform under (local) gauge transformation.

by SW map. Interestingly, one can also now carry out quantum mechanical analysis in first quantized formalism from the Schrödinger equation derived from this effective theory. The main motivation for carrying out this investigation in NR framework is that here the transition from second quantization to first quantization is rather quite straightforward, and infact first and second quantized formalism are completely equivalent as far as Galilean invariant models are concerned. It may be recalled that there is no particle production in a Galilean invariant field theory. Also, an N-particle state can be constructed by superposing in terms of first quantized N-particle wave functions, the states obtained by N-fold actions of the creation operators on the vacuum. Thus, if one restricts the N-particle sector, while quantizing canonically, one recovers the first quantized N-particle wave functions. So although the noncommutative $\hat{\psi}$ field in Schrödinger equation on a plane can have an interpretation of probability amplitude, it is not clear that this feature will persist with the SW field ψ when an effective commutative theory is obtained from the original noncommutative theory through the use of SW map. We find in this chapter that unless the gauge field configuration is such that the corresponding magnetic field is constant, the probabilistic interpretation will not go through. This indicates that the nature of the gauge field must be of "background" type, rather than a dynamical one. As in the relativistic case, we shall analyse this problem by writing down an effective theory of the original noncommutative Schrödinger action coupled to background $U(1)_\star$ gauge theory in terms of usual commutative variables by using SW map. After setting up the formalism, we identify the physical variables by proper "renormalisation" of wave function and mass to identify the probability current appropriate for the first quantized formalism.

Finally, as an example, we take up the case of Hall conductivity in noncommutative plane. In this context, we would like to point out an important aspect of this noncommutativity which has its deep connection with Quantum Hall systems [67]. Lots of authors have made quite an extensive study of this deep connection [68]–[77]. To start with, the simple problem of Landau level and Hall conductivity in noncommutative plane was addressed by a number of authors [30, 78, 79, 80, 81]. However, the results of various authors do not seem to be convergent on the issue of effect of the noncommutative parameter θ on Hall conductivity; some show deviations and others show no deviations from usual commutative theory. Note that these analysis and their subsequent results involve noncommutative electric and magnetic fields, which in general are not gauge invariant objects even for the simplest $U(1)_\star$ gauge group; they rather transform covariantly. Consequently they cannot correspond to any observables in a generic case. This limitation can be avoided, for example, by writing an effective theory in ordinary commutative space by making use of SW map [14] and compute Hall conductivity in terms of the usual $U(1)$ gauge invariant electric and magnetic fields [82]. This will clearly open another avenue to compare with the existing results in the literature. Here, we would also like to mention that in a recent paper [83], the authors also have analysed this problem by using a modified norm–preserving unitarised SW map and have studied the effect of noncommutativity in Hall systems apart from Aharanov–Bohm effect. In contrast, in this chapter we apply the usual SW map to construct an effective commutative theory and identify the probability current after wave function and mass renormalisation. This in turn, is used to compute the Hall conductivity.

4.1 The Seiberg-Witten map

The SW map has been an important ingredient in the analysis of noncommutative quantum field theories. The rational behind this map derives from the observation that commutative and noncommutative field theories result from different regularizations of the same gauge theory, at least in two dimensions. Thus, a map should exist between these theories which reflects the fact that the physical content of the two theories is the same. In this section we shall present a brief review of this celebrated map [14] which has played a very important role in the study of noncommutative quantum field theory and will also play a significant role in the rest of this chapter.

It is an explicit map connecting a given noncommutative gauge theory with a conventional gauge theory. Let us consider the case in which the noncommutative gauge theory is governed by a Yang-Mills (YM) Lagrangian for the gauge potential $\hat{A}_\mu$, transforming under gauge transformations according to

$$\hat{\delta}_{\hat{\varepsilon}} A_\mu(x) \quad = \quad \hat{A}'_\mu(x) - \hat{A}_\mu(x) = D_\mu[\hat{A}]\hat{\varepsilon}(x) \ . \tag{4.1}$$

The SW map connects the noncommutative YM Lagrangian to some unconventional Lagrangian on the commutative side. What is conventional in the latter, apart from the fact that fields are multiplied with the ordinary product is that the transformation law for the gauge field A_μ is governed by the ordinary covariant derivative:

$$\delta_\varepsilon A_\mu(x) = A'_\mu(x) - A_\mu(x) = D_\mu[A]\varepsilon(x). \tag{4.2}$$

Note that we are calling $\hat{\varepsilon}$, the infinitesimal gauge transformation parameter in the noncommutative theory to distinguish it from ε, its mapped counterpart in the ordinary theory. Hence, the mapping should include, apart from a connection between $\hat{A}_\mu$ and A_μ, one for connecting $\hat{\varepsilon}$ and ε.

It turns out that the equivalence holds at the level of orbit space, the physical configuration space of gauge theories. This means that if two gauge fields $\hat{A}_\mu$ and $\hat{A}'_\mu$ belonging to the same orbit can be connected by a noncommutative gauge transformation $\exp_*(i\hat{\varepsilon})$, then A'_μ and A_μ, the corresponding mapped gauge fields will also be gauge equivalent by an ordinary gauge transformation $\exp(i\varepsilon)$. An important point is that the mapping between $\hat{\varepsilon}$ and ε necessarily depends on A_μ. Indeed, if $\hat{\varepsilon}$ were a function of ε solely, the ordinary and the noncommutative gauge groups would be identical. That this is not possible can be seen just by considering the case of a $U(1)$ gauge theory in which, through a redefinition of the gauge parameter, one would be establishing an isomorphism between noncommutative $U_*(1)$ and commutative $U(1)$ gauge groups.

Then, the SW mapping consists in finding

$$\begin{aligned} \hat{A} &= \hat{A}[A;\theta] \\ \hat{\varepsilon} &= \hat{\varepsilon}[\varepsilon, A;\theta] \end{aligned} \tag{4.3}$$

so that the equivalence between orbits holds

$$\hat{A}[A] + \hat{\delta}_{\hat{\varepsilon}}\hat{A}[A] = \hat{A}[A + \delta_\varepsilon A]. \tag{4.4}$$

Using the explicit form of gauge transformations and expanding to first order in $\theta = \delta\theta$, the solution of eq.(4.4) reads:

$$\hat{A}_\mu[A] = A_\mu - \frac{1}{4}\delta\theta^{\rho\sigma}\{A_\rho, \partial_\sigma A_\mu + F_{\sigma\mu}\} + O(\delta\theta^2)$$

$$\hat{\varepsilon}[\varepsilon, A] = \varepsilon + \frac{1}{4}\delta\theta^{\rho\sigma}[\partial_\rho\varepsilon, A_\sigma\} + O(\delta\theta^2) \tag{4.5}$$

where the products on the right hand side, such as $\{A_\rho, \partial_\sigma A_\mu\} = A_\rho.\partial_\sigma A_\mu + \partial_\sigma A_\mu.A_\rho$ are ordinary matrix products.

Concerning the field strength, the connection is given by:

$$\hat{F}_{\mu\nu}[A] = F_{\mu\nu} + \frac{1}{4}\delta\theta^{\alpha\beta}\left(2\{F_{\mu\alpha}, F_{\nu\beta}\} - \{A_\alpha, D_\beta F_{\mu\nu} + \partial_\beta F_{\mu\nu}\}\right) + O(\delta\theta^2). \tag{4.6}$$

One can interpret these equations as differential equations describing the passage from A_μ^θ (the gauge field in a theory with parameter θ) to $A_\mu^{\theta+\delta\theta}$ (the gauge field in a theory with parameter $\theta + \delta\theta$). Integrating these equations leads to the passage from $L_{YM}[\hat{A}]$ (the noncommutative version of YM Lagrangian), to $L[A, \theta]$ which is a complicated but commutative equivalent Lagrangian to all orders in θ.

4.2 U(1)$_\star$ gauge invariant Schrödinger action

We start with the action of a Schrödinger field ψ coupled with $U(1)$ background gauge field $A_\mu(x)$ in the ordinary commutative space

$$S = \int d^3x \psi^\dagger (iD_0 + \frac{1}{2m} D_i D_i)\psi \tag{4.7}$$

where, $D_\mu = (\partial_\mu - igA_\mu)$ is the covariant derivative operator and g is the coupling constant. The corresponding $U(1)_\star$ gauge invariant action in noncommutative space is

$$\hat{S} = \int d^3x \hat{\psi}^\dagger \star (i\hat{D}_0 + \frac{1}{2m}\hat{D}_i \star \hat{D}_i) \star \hat{\psi} \tag{4.8}$$

where the caret notation indicates noncommutative nature of the variables $\hat{\psi}$ (assumed to be Schwartzian [23]) which compose through the star product (introduced earlier in eqn.(2.21)) defined as

$$\left(\hat{f} \star \hat{g}\right)(x) = e^{\frac{i}{2}\theta^{\alpha\beta}\partial_\alpha\partial'_\beta} \hat{f}(x)\hat{g}(x')|_{x'=x} \ . \tag{4.9}$$

Under $\star$ composition the Moyal bracket between the coordinates is

$$[\hat{x}^\mu, \hat{x}^\nu]_\star = i\theta^{\mu\nu} \tag{4.10}$$

which is isomorphic to the algebra of operator valued coordinates in noncommutative space $\left[x_{op}^{\mu}, x_{op}^{\nu}\right] = i\theta^{\mu\nu}$. Also $\left(\hat{D}_{\mu}\star = \partial_{\mu} - ig\hat{A}_{\mu}\star\right)$ is the appropriate covariant derivative operator in noncommutative space. Under the simultaneous $U(1)_{\star}$ gauge transformation

$$\hat{\psi}(x) \mapsto \hat{\psi}'(x) = \hat{U}(x) \star \hat{\psi}(x) \tag{4.11}$$

$$\hat{A}_{\mu}(x) \mapsto \hat{A}'_{\mu}(x) = \hat{U}(x) \star \hat{A}_{\mu}(x) \star \hat{U}^{\dagger}(x) + \frac{i}{g}\hat{U}(x) \star \partial_{\mu}\hat{U}^{\dagger}(x) \tag{4.12}$$

where $\hat{U}(x)$ is the star unitary function satisfying

$$\hat{U}(x) \star \hat{U}^{\dagger}(x) = \hat{U}^{\dagger}(x) \star \hat{U}(x) = 1 \tag{4.13}$$

one can show that $\left(\hat{D}_{\mu} \star \hat{\psi}\right) \to \left(\hat{D}'_{\mu} \star \hat{\psi}'\right) = \hat{U}(x)\star\left(\hat{D}_{\mu} \star \hat{\psi}\right)$, i.e it transforms covariantly. Note that $\hat{U}^{\dagger}(x)$ is not equal to $\hat{U}^{-1}(x)$ unless $\hat{U}(x) \in U(1)_{\star}$-the rank 1 gauge group.
The equation of motion for the fundamental field $\hat{\psi}(x)$ is

$$(i\hat{D}_0 + \frac{1}{2m}\hat{D}_i \star \hat{D}_i) \star \hat{\psi} = 0. \tag{4.14}$$

The usual $\star$–gauge invariant matter or probability current density $\hat{j}_{\mu}$ following from eq.(4.14) is given by:

$$\hat{j}_0 = \hat{\rho} = \hat{\psi}^{\dagger} \star \hat{\psi} \tag{4.15}$$

$$\hat{j}_i = \frac{1}{2mi}\left[\hat{\psi}^{\dagger} \star \left(\hat{D}_i \star \hat{\psi}\right) - \left(\hat{D}_i \star \hat{\psi}\right)^{\dagger} \star \hat{\psi}\right]; \quad (i = 1, 2) \tag{4.16}$$

which satisfy the usual continuity equation

$$\partial_t\hat{j}_0 + \partial_i\hat{j}_i = 0. \tag{4.17}$$

Here, we would like to mention that $\hat{j}_0$ is not manifestly positive definite. However, it can be made so by modifying it by adding a suitable total divergence term, so that $\hat{j}_0$ (upto a divergence term) can be regarded as a probability density and corresponding $\hat{j}_i$'s as probability currents when we switch over to first quantized version from the second quantized one. One can at this stage add a $\star$–gauge invariant dynamical term $-\frac{1}{4}\int d^nx\hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu}$ to the action (4.8) where the field strength $\hat{F}_{\mu\nu}$ is defined as $\hat{F}_{\mu\nu} = \frac{i}{g}\left[\hat{D}_\mu, \hat{D}_\nu\right]_\star = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - ig\left[\hat{A}_\mu, \hat{A}_\nu\right]_\star$, and identify a $U(1)_\star$ charge current density $\hat{J}^\mu$ through the equation of motion for the $\hat{A}_\mu$ field $\tilde{D}_\nu \star \hat{F}^{\mu\nu} = \hat{J}^\mu$ where $\tilde{D}_\mu \star := \partial_\mu - ig\left[\hat{A}_\mu, \quad \right]_\star$. The explicit form of $\hat{J}_\mu$ is given by:

$$\hat{J}_0 = g\hat{\psi} \star \hat{\psi}^\dagger \tag{4.18}$$

$$\hat{J}_i = \frac{g}{2mi}\left[\left(\hat{D}_i \star \hat{\psi}\right) \star \hat{\psi}^\dagger - \hat{\psi} \star \left(\hat{D}_i \star \hat{\psi}\right)^\dagger\right]; \quad (i = 1, 2) \tag{4.19}$$

Unlike $\hat{j}_\mu$, $\hat{J}_\mu$ are not $U(1)_\star$ gauge invariant, rather they transform covariantly and satisfy a covariant version of continuity equation $\hat{D}_0\hat{J}_0 + \hat{D}_i\hat{J}_i = 0$. After identifying $\hat{J}_\mu$, we can do away with the dynamical term and deal with the Galilean invariant action (4.8) itself. Note that similar covariant transformation property holds for $\hat{F}_{\mu\nu}$, i.e. $\hat{F}_{\mu\nu} \mapsto \hat{F}'_{\mu\nu} = \hat{U}(x) \star \hat{F}_{\mu\nu} \star \hat{U}^\dagger(x)$. This is reminescent of what happens in Yang-Mills theory. Consequently, a generic configuration for $\hat{F}_{\mu\nu}$ (except for the special case of $\hat{F}_{\mu\nu} =$ constant) does not remain $U(1)_\star$ gauge invariant. $\hat{F}_{\mu\nu}$ therefore, does not correspond to an observable. A $U(1)_\star$ gauge invariant noncommutative Chern-Simons action $\hat{S}_{cs} \sim \int d^3x \epsilon^{\mu\nu\lambda}\left\{\hat{A}_\mu \star \partial_\nu \hat{A}_\lambda + \frac{2i}{3}\hat{A}_\mu \star \hat{A}_\nu \star \hat{A}_\lambda\right\}$ could also be added to eq.(4.8), instead of the noncommutative Maxwell term, as this dynamical term is not associated with any "photon" and can be coupled to NR matter fields without apparently spoiling the Galilean symmetry, if there is no spacetime noncommutativity ($\theta^{0i} = 0$).

4.3 Effective Theory constructed in commutative space

We now move on to construct an effective action starting from eq.(4.8) by using the SW map in the lowest order in $\theta^{\mu\nu}$ [59]:

$$\hat{\psi} = \psi - \frac{1}{2}\theta^{mj} A_m \partial_j \psi \tag{4.20}$$

$$\hat{A}_i = A_i - \frac{1}{2}\theta^{mj} A_m \left(\partial_j A_i + F_{ji}\right). \tag{4.21}$$

Taking $\theta^{0i} = 0$, we substitute the above form of $\hat{\psi}$ and $\hat{A}_\mu$ given by eq.(s) (4.20) and (4.21) in the action (4.8). After some algebra one finds the following usual $U(1)$ gauge invariant expression for the effective action.

$$\hat{S} \stackrel{\text{SW map}}{=} \int d^3x \left[\left(1-\frac{\theta B}{2}\right)\left(\psi^\dagger iD_0\psi\right)+\frac{i}{2}\theta^{mj}\left(\psi^\dagger D_j\psi\right)F_{m0}\right.$$
$$+\frac{1}{2m}\left(1-\frac{\theta B}{2}\right)\left(\psi^\dagger D_iD_i\psi\right)$$
$$\left.+\frac{1}{2m}\theta^{mj}\left(\psi^\dagger D_iD_j\psi\right)F_{mi}+\frac{1}{4m}\theta^{mj}\left(\psi^\dagger D_j\psi\right)\partial_iF_{mi}\right]. \tag{4.22}$$

The third and fourth terms in the paranthesis can now be combined using the relation $F_{mi} = B\epsilon_{mi}$ to get

$$\hat{S} \stackrel{\text{SW map}}{=} \int d^3x \left[\left(1-\frac{\theta B}{2}\right)\left(\psi^\dagger iD_0\psi\right)+\frac{i}{2}\theta^{mj}\left(\psi^\dagger D_j\psi\right)F_{m0}\right.$$
$$\left.+\frac{1}{2m}\left(1+\frac{\theta B}{2}\right)\left(\psi^\dagger D_iD_i\psi\right)+\frac{1}{4m}\theta^{mj}\left(\psi^\dagger D_j\psi\right)\partial_iF_{mi}\right]. \tag{4.23}$$

A hermitian form of this action can easily be written by dropping certain boundary terms to get

$$\hat{S} = \int d^3x \left[\left(1-\frac{\theta B}{2}\right)\left(\frac{i}{2}\psi^\dagger \overleftrightarrow{D}_0 \psi\right)-\frac{1}{2m}\left(1+\frac{\theta B}{2}\right)(D_i\psi)^\dagger(D_i\psi)+\frac{i}{4}\theta^{mj}(\psi^\dagger \overleftrightarrow{D}_j \psi)F_{m0}\right.$$
$$\left.+\frac{1}{8m}\theta^{mj}\left(\psi^\dagger \overleftrightarrow{D}_j \psi\right)\partial_iF_{mi}+...\right] \tag{4.24}$$

where the dots indicating missing terms, involving $\partial_\mu F_{\nu\lambda}$, have not been written down explicitly, as they play no role in the simplectic structure of the theory. These terms represent additional possible interactions. Note that this action is not in the canonical form. As a result, the field ψ in second quantized formalism does not have a canonical structure for the equal time commutation relation between ψ and $\psi^\dagger$[3]:

$$\left[\psi(x),\psi^\dagger(y)\right]=\left(1+\frac{\theta B}{2}\right)\delta^2(x-y). \tag{4.25}$$

Note that this commutator is easily obtained by elevating the DB between ψ and $\psi^\dagger$ given as

$$\{\psi(x),\psi^\dagger(y)\}_{DB}=-i(1+\frac{\theta B}{2})\delta^2(x-y) \tag{4.26}$$

[3]In this section, we use the same notation $\psi^\dagger(x)$ to indicate complex (hermitian) conjugate of ψ at the classical (quantum) level. Also the operator nature of $\psi(x)$ and $\psi^\dagger(x)$ at the quantum level is not displayed explicitly by putting a caret on the top; the caret is now reserved to indicate noncommutative variables. This, expectedly, will not give rise to any confusion as their respective nature should be clear from the context itself.

which in turn is obtained by strong imposition of the following pair $(\Lambda_a; (a = 1, 2))$ of second class constraints

$$\Lambda_1(x) = \Lambda_2^*(x) = \Pi_\psi(x) - \frac{i}{2}\left(1 - \frac{\theta B}{2}\right)\psi^\dagger(x) \approx 0 \tag{4.27}$$

where, Π_ψ and $\Pi_{\psi^\dagger}(= (\Pi_\psi)^\dagger)$ are the canonically conjugate momenta to ψ and $\psi^\dagger$ respectively. Since A_μ's are background gauge fields, they are not included in the configuration space. So we must have $[A_\mu(x), \psi(y)] = 0$. This non-standard form of the commutation relation (4.25) indicates that ψ cannot represent the basic field variable or the wave-function in the corresponding first quantized formalism. This is further re-inforced by the observation that for the generic case, where B has an x-dependence, the Euler–Lagrange equation for $\psi^\dagger$, following from eq.(4.23)

$$\begin{aligned}\left(1 - \frac{\theta B}{2}\right) iD_0\psi + \frac{1}{2m}\left(1 + \frac{\theta B}{2}\right) D_iD_i\psi + \frac{i}{2}\theta^{mj}\left(D_j\psi\right)F_{m0}& \\ + \frac{1}{4m}\theta^{mj}\left(D_j\psi\right)\partial_iF_{mi} = 0&\end{aligned} \tag{4.28}$$

can only be brought *almost* to the form of standard Schrödinger equation

$$iD_0\psi + \frac{1}{2\tilde{m}}D_iD_i\psi + \frac{i}{2}\theta^{mj}\left(D_j\psi\right)F_{m0} + \frac{1}{4m}\theta^{mj}\left(D_j\psi\right)\partial_iF_{mi} = 0 \tag{4.29}$$

for the first pair of terms by introducing a non-constant $\tilde{m}$ as

$$\tilde{m} = (1 - \theta B)\, m. \tag{4.30}$$

To identify the basic field variable, let us scale ψ as

$$\psi \mapsto \tilde{\psi} = \sqrt{1 - \frac{\theta B}{2}}\psi \tag{4.31}$$

so that the commutation relation (4.25) can be cast as

$$\left[\tilde{\psi}(x), \tilde{\psi}^\dagger(y)\right] = \delta^2(x - y) \tag{4.32}$$

and $\tilde{\psi}$ and $\tilde{\psi}^\dagger$ can now be interpreted as annihilation and creation operators in second quantized formalism. Let us now construct $|x\rangle$ (the state corresponding to a single particle located at x)

by the action of this creation operator acting on the normalised vacuum state $|0\rangle$ ($\langle 0|0\rangle = 1$) as $|x\rangle = \tilde{\psi}^{\dagger}(x)|0\rangle$, so that the standard inner product relation $\langle y|x\rangle = \delta^{(2)}(x-y)$ and the resolution of identity ($1 = \int d^2x|x\rangle\langle x|$) holds. Now writing an arbitrary one-particle state $|\tilde{\psi}\rangle = \int d^2x\tilde{\psi}(x)|x\rangle$ in terms of wave function $\tilde{\psi}(x) = \langle x|\tilde{\psi}\rangle$ corresponding to first quantized formalism, one can easily see that the normalisation condition

$$\int d^2x\tilde{\psi}^{\dagger}\tilde{\psi} = 1 \tag{4.33}$$

follows trivially by demanding $\langle\tilde{\psi}|\tilde{\psi}\rangle = 1$. So this transition from second quantized to first quantized formalism clearly shows that it is $\tilde{\psi}$, rather than ψ itself, which corresponds to the normalised wave-function or the basic field variable in the action. It is therefore desirable to re-express the action (4.23) in terms of $\tilde{\psi}$ and ensure that it is in the standard form in the first pair of terms in both of these expressions. Clearly this can be done only for a constant B-field[4]. Note that with this, $\tilde{m}$ (eq.(4.30)) also becomes constant. Such a constant magnetic field can only arise from an appropriate background gauge field. In presence of a dynamical term, like Chern–Simons action, B cannot be ensured to be a constant and consequently the Schrödinger equation describing the time evolution of the normalised wave-function in terms of the "renormalised" SW field $\tilde{\psi}$ cannot be obtained. In rest of the chapter, we shall therefore consider a constant background for field strength tensor $F_{\mu\nu}$[5]. In this case, the above action (4.24) should be written in terms of $\tilde{m}$ (eq.(4.30)), $\tilde{\psi}$ (eq.(4.31)) to get a canonical form for the Schrödinger action

$$\hat{S} \stackrel{\mathrm{SW\,map}}{=} \int d^3x \left[\left(\tilde{\psi}^{\dagger}iD_0\tilde{\psi}\right) + \frac{1}{2\tilde{m}}\left(\tilde{\psi}^{\dagger}D_iD_i\tilde{\psi}\right) + \frac{i}{2}\theta^{mj}\left(\tilde{\psi}^{\dagger}D_j\tilde{\psi}\right)F_{m0}\right]. \tag{4.34}$$

The field $\tilde{\psi}$ and mass parameter $\tilde{m}$ can now be regarded as renormalised wave-function and mass respectively. We shall therefore treat $\tilde{\psi}$ (and not ψ) as the basic field in our theory.

[4]In addition, if the electric field is also taken to be constant, then the additional interaction terms in eq.(4.24) will vanish thus yielding the simplest possible action incorporating noncommutativity.

[5]Since we are considering a constant background magnetic field, it will be advantegeous to consider a constant electric field background also. Under SW map, a constant configuration of $F_{\mu\nu}$ results in a constant $\hat{F}_{\mu\nu}$ and vice-versa ($\hat{F}_{\mu\nu} = F_{\mu\nu} - \theta^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}$ and $F_{\mu\nu} = \hat{F}_{\mu\nu} + \theta^{\alpha\beta}\hat{F}_{\mu\alpha}\hat{F}_{\nu\beta}$). Also with this constant configuration, all the missing terms in eq.(4.24) vanish.

This gives the effect of noncommutativity in the observed mass $\tilde{m}$. A point which is worth mentioning is that the expression for $\tilde{m}$ (eq.(4.30)) indicates that the external magnetic field B has a critical value $B_c = \frac{1}{\theta}$. Clearly, for $B > B_c$, $\tilde{m}$ becomes negative which is unphysical. We shall see later in chapter 5, how one can define physical quantities at this critical point B_c and beyond it. Using eq.(4.30), one can easily see that the ratio of the observed masses $\tilde{m}_1$ and $\tilde{m}_2$ corresponding to two distinct magnetic fields B_1 and B_2, satisfies (upto order θ) $\frac{\tilde{m}_1}{\tilde{m}_2} = 1 - \theta\left(B_1 - B_2\right)$ which in turn, can be used to get an estimate for noncommutative parameter θ. Incidentally, this relation (4.30) was also obtained earlier by Duval et.al [78].
The equation of motion for the fundamental field $\tilde{\psi}$ (from the action (4.34)) is

$$K\tilde{\psi} = 0 \tag{4.35}$$

where, K is the operator given by:

$$K = iD_0 + \frac{1}{2\tilde{m}}D_iD_i + \frac{i}{2}\theta^{mj}F_{m0}D_j. \tag{4.36}$$

It is easy to verify

$$-i\left(\tilde{\psi}^\dagger K\tilde{\psi} - (K\tilde{\psi})^\dagger\tilde{\psi}\right) = \partial_\mu j_\mu \tag{4.37}$$

where, the $3(= 1 + 2)$-currents j_μ are given by:

$$j_0 = \tilde{\psi}^\dagger\tilde{\psi} + \frac{i}{2}\theta^{mj}\left(D_m\tilde{\psi}\right)^\dagger\left(D_j\tilde{\psi}\right) \tag{4.38}$$

$$\begin{aligned} j_i \;&=\; \frac{1}{2\tilde{m}i}\left[\left\{\tilde{\psi}^\dagger\left(D_i\tilde{\psi}\right) - \left(D_i\tilde{\psi}\right)^\dagger\tilde{\psi}\right\}\right.\\ &\quad\left. + \frac{i}{2}\theta^{mj}\left\{\left(D_m\tilde{\psi}\right)^\dagger\left(D_iD_j\tilde{\psi}\right) + \left(D_iD_j\tilde{\psi}\right)^\dagger\left(D_m\tilde{\psi}\right)\right\}\right]. \end{aligned} \tag{4.39}$$

Using eq.(s) (4.35) and (4.37), we find that the continuity equation is automatically satisfied by j_μ, therefore one is tempted to identify j_μ (eq.(s) (4.38, 4.39)) as the probability density and probability current of the system. But as it turns out that the probability density and currents have to be determined from $\hat{j}_\mu$ (eq.(s) (4.15,4.16)) as the components of this current played the role of gauge invariant probability density and probability current in noncommutative formulation (see section 4.2). All that we have to do here is to apply SW map to rewrite $\hat{j}_\mu$ in terms of field ψ and then in terms of the renormalised field $\tilde{\psi}$ (eq.(4.31)). At this stage, one can note an interesting fact that $\hat{j}_\mu$ also has the same form as that of j_μ (eq.(s) (4.38, 4.39) except that one has to just replace $\tilde{\psi}$ by ψ :

$$\hat{j}_0 = \psi^\dagger \psi + \frac{i}{2}\theta^{mj} \left(D_m \psi\right)^\dagger \left(D_j \psi\right) \tag{4.40}$$

$$\begin{aligned}\hat{j}_i &= \frac{1}{2\tilde{m}i}\left[\left\{\psi^\dagger \left(D_i \psi\right) - \left(D_i \psi\right)^\dagger \psi\right\}\right. \\ &\left. +\frac{i}{2}\theta^{mj}\left\{\left(D_m \tilde{\psi}\right)^\dagger \left(D_i D_j \tilde{\psi}\right) + \left(D_i D_j \tilde{\psi}\right)^\dagger \left(D_m \tilde{\psi}\right)\right\}\right]\end{aligned} \tag{4.41}$$

so that $\hat{j}_\mu$ and j_μ are related by $j_\mu = (1 - \frac{\theta B}{2})\hat{j}_\mu$ as follows from eq.(4.31) and the fact that the currents are bilinear in their respective fields. This is not surprising as ψ also satisfies (4.35) ($K\psi = 0$) upto order θ. However, note that $\hat{j}_0$ (eq.(4.40)), does not have the standard form because of the presence of the θ-dependent term. Not only that, it is not manifestly positive-definite point-wise. Consequently, there is a difficulty in identifying $\hat{j}_0$ as the probability density directly in the "first quantized" version of single-particle quantum mechanics. This problem can be easily seen to be, however, inherited from the original noncommutative formulation itself. For that recall, this problem was avoided there by modifying $\hat{j}_0$ (eq.(4.15)) by a total divergence term to isolate a positive definite quantity to be identified as the probability density. Following the same methodology here, we note that $\hat{j}_0$ (eq.(4.40)) can also be brought to almost standard form upto a $\left(1 - \frac{\theta B}{2}\right)$ factor (assuming to be positive) by dropping a total divergence term, so that we have

$$\int d^2x \hat{j}_0 = \left(1 - \frac{\theta B}{2}\right) \int d^2x \psi^\dagger \psi \tag{4.42}$$

which however takes the canonical form

$$\int d^2x \hat{j}_0 = \int d^2x \tilde{\psi}^\dagger \tilde{\psi} \tag{4.43}$$

when rewritten in terms of renormalised wave-function $\tilde{\psi}$ (eq.(4.31)). With the normalisation condition (4.33), it now becomes clear that it is $\tilde{\psi}^\dagger \tilde{\psi}$ (or $\hat{j}_0$ upto a total divergence term) has now to be identified as the probability density which is manifestly positive definite at all points[6]. It immediately follows that the spatial components of $\hat{j}_\mu$, i.e $\hat{j}_i$ *must* correspond to the spatial component of the probability current, as $\hat{j}_\mu$ satisfies the continuity equation $\partial_\mu \hat{j}_\mu = 0$. Therefore the particle current (for a single particle) $I_i^{(1)}$ in the i-th direction is obtained by integrating $\hat{j}_i$ over the variable in the orthogonal direction, i.e $I_1^{(1)} = \int dx^2 \hat{j}_1$ and $I_2^{(1)} = \int dx^1 \hat{j}_2$. We shall however be interested in the transverse current $I_2^{(1)}$ in section 4.5, as the longitudinal current $I_1^{(1)}$ will vanish.

4.4 Galilean symmetry generators

In this section we shall try to construct all the Galilean symmetry generators for the model defined by the action (4.34) where $\tilde{\psi}$ is taken to be the basic field and F_{m0} representing the constant electric field E_m in the background. The corresponding gauge field A_μ is therefore not included in the configuration space variable. Before we start carrying out the Hamiltonian analysis, we must ensure that the action is in a manifestly hermitian form. We therefore rewrite the action (4.24) in terms of $\tilde{\psi}$ (eq.(4.31)) as

$$\hat{S} = \int d^3x \left[(\frac{i}{2}\tilde{\psi}^\dagger \overleftrightarrow{D}_0 \tilde{\psi}) - \frac{1}{2\tilde{m}}(D_i\tilde{\psi})^\dagger (D_i\tilde{\psi}) + \frac{i}{4}\theta^{mj}\{(\tilde{\psi}^\dagger \overleftrightarrow{D}_j \tilde{\psi})E_m \right]. \tag{4.44}$$

Coming to the symplectic structure, the conjugate momenta corresponding to the configuration space variables are

$$\Pi_{\tilde{\psi}} = \frac{i}{2}\tilde{\psi}^\dagger \qquad , \qquad \Pi_{\tilde{\psi}^\dagger} = -\frac{i}{2}\tilde{\psi}. \tag{4.45}$$

The canonical Hamiltonian density can be calculated by a Legendre transform which in turn can be integrated to get the Hamiltonian as

$$H = \int d^2x \left[\frac{1}{2\tilde{m}}(D_i\tilde{\psi})^\dagger (D_i\tilde{\psi}) - \frac{i}{4}\theta^{mj}\{(\tilde{\psi}^\dagger \overleftrightarrow{D}_j \tilde{\psi})E_m - A_0(\tilde{\psi}^\dagger\tilde{\psi}) \right]. \tag{4.46}$$

[6] Note that this technique is quite common in quantum field theory. In this context, it may be recalled that the Noether's expression of energy-momentum tensor (say in free Maxwell theory in 3+1-dimension), which is nothing but the density and current of conserved energy-momentum four-vector, is amended by a four divergence term to render it symmetric and gauge invariant (Belinfante method). So here too the original $\hat{j}_0$ is modified by dropping a total divergence term at the field theoretic level to render it positive definite so that it is interpretable as probability density when we switch over to "first quantized" version of quantum mechanics.

It is clear from eq.(4.45) that the system contains second-class constraints which can be strongly implemented by Dirac scheme to obtain the following bracket

$$\left\{\tilde{\psi}(x), \tilde{\psi}^{\dagger}(y)\right\} = -i\delta^2(x-y) \tag{4.47}$$

which in turn can be elevated to obtain the quantum commutator (4.32). Note that this bracket can also be obtained, in fact more simply by using Faddeev–Jackiw (FJ) approach as this Lagrangian (4.44) is first order in time derivative. A quick and easy calculation (using eq.(4.32)) shows that the above Hamiltonian (4.46) generates appropriate time translation

$$\dot{\psi}(x) = \{\psi(x), H\}. \tag{4.48}$$

The generator of spatial translation and $SO(2)$ rotation can now be easily constructed using Noether's theorem to get

$$P_i = \int d^2x \left[\Pi_{\tilde{\psi}}\partial_i\tilde{\psi}(x) + \Pi_{\tilde{\psi}^\dagger}\partial_i\tilde{\psi}^\dagger(x)\right] = \int d^2x \frac{i}{2}\tilde{\psi}^\dagger(x) \overleftrightarrow{\partial_i} \tilde{\psi}(x) \tag{4.49}$$

$$J = \frac{i}{2}\int d^2x \epsilon_{ij} x_i \tilde{\psi}^\dagger(x) \overleftrightarrow{\partial_j} \tilde{\psi}(x) \tag{4.50}$$

which generates appropriate translation and rotation[7]:

$$\left\{\tilde{\psi}(x), P_i\right\} = \partial_i\tilde{\psi}(x) \tag{4.51}$$

$$\left\{\tilde{\psi}(x), J\right\} = \epsilon_{ij} x_i \partial_j \tilde{\psi}(x). \tag{4.52}$$

Note that J (eq.(4.50)) consists of only the orbital part of the angular momentum as in our simplistic treatment we have ignored the spin degree of freedom for the field $\tilde{\psi}$, so that it transforms as an $SO(2)$ scalar. Using the DB (4.47), one can verify the following algebra:

$$\begin{aligned} \{P_i, P_j\} = \{P_i, H\} = \{J, H\} = 0 \\ \{P_k, J\} = \epsilon_{kl} P_l. \end{aligned} \tag{4.53}$$

[7]The adjective "appropriate" in this context means the brackets $\{\Phi(x), \mathcal{G}\}$ are just equal to the Lie derivative $[\mathcal{L}_{V_\mathcal{G}}(\Phi(x))]$ of a generic field $\Phi(x)$ with respect to the vector field $V_\mathcal{G}$, associated with the symmetry generator $\mathcal{G}$. We have not, of course, displayed any indices here. The field $\Phi(x)$ may be a scalar, spinor, vector or tensor field in general. In this case, it corresponds to the field $\tilde{\psi}(x)$ and not A_μ as it is a background field. And $\mathcal{G}$ can be, for example, the momentum (P_i) or angular momentum (J) operator generating translation and spatial rotation, respectively. The associated vector fields $V_\mathcal{G}$ are thus given as ∂_i and ∂_ϕ, respectively (ϕ being the angle variable in the polar coordinate system in the two-dimensional plane).

This shows that P_k and J form a closed $E(2)$ (Euclidian) algebra. Now coming to the boost, we shall try to analyse the system from first principle and shall check the covariance of eq.(4.29) under Galileo boost. For this, we essentially follow [84]. To that end, we consider an infinitesimal Galileo boost along the X-direction,

$$t \mapsto t' = t, \quad x^1 \mapsto x^{1\prime} = x^1 - vt, \quad x^2 \mapsto x^{2\prime} = x^2 \tag{4.54}$$

with an infinitesimal velocity parameter "v". Notwithstanding the fact that Galilean spacetime $\mathcal{M}$ does not have a metric, one can define tangent space $T_p(\mathcal{M})$ or its dual cotangent space $T_p^\star(\mathcal{M})$ on any point $p \in \mathcal{M}$. The canonical basis of $T_p(\mathcal{M})$ corresponding to unprimed and primed frames are thus given as $(\partial/\partial t, \partial/\partial x^i)$ and $(\partial/\partial t', \partial/\partial x^{i\prime})$, respectively. They are related as

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x^1}, \quad \frac{\partial}{\partial x^{i\prime}} = \frac{\partial}{\partial x^i} \ . \tag{4.55}$$

As for the transformation properties of the basic fields are concerned, we note that in the first quantized version $\tilde{\psi}$ is going to represent probability amplitude and $\tilde{\psi}^\dagger\tilde{\psi}$ represents the probability density. Hence in order that $\tilde{\psi}^\dagger\tilde{\psi}$ remains invariant under Galileo boost ($\tilde{\psi}'^\dagger(x', t')\tilde{\psi}'(x', t') = \tilde{\psi}^\dagger(x, t)\tilde{\psi}(x, t)$), we expect $\tilde{\psi}$ to change atmost by a phase factor. This motivates us to make the following ansatz :

$$\tilde{\psi}(x, t) \mapsto \tilde{\psi}'(x', t') = e^{iv\eta(x,t)}\tilde{\psi}(x, t) \simeq (1 + iv\eta(x, t))\tilde{\psi}(x, t)) \tag{4.56}$$

for the transformation of the field $\tilde{\psi}$ under infinitesimal Galileo boost ($v << 1$). As far as the transformation properties of the gauge field $A_\mu(x)$ is concerned, it should transform like the basis $\frac{\partial}{\partial x^\mu}$ (eq.(4.55)) of $T_p(\mathcal{M})$. This is because $A_\mu(x)$'s can be regarded as the components of the one-form $A(x) = A_\mu(x)dx^\mu \in T_p^\star(\mathcal{M})$. It thus follows that

$$\begin{aligned} A_0(x) \mapsto {A_0}'(x') &= A_0(x) + vA_1(x) \\ A_i(x) \mapsto {A_i}'(x') &= A_i(x) \end{aligned} \tag{4.57}$$

under Galileo boost. Now demanding that the action (4.34) remains invariant or equivalently the equation of motion (4.35, 4.36) remains covariant implies that the following pair of equations

$$iD_0\tilde{\psi} + \frac{1}{2\tilde{m}}D_iD_i\tilde{\psi} + \frac{i}{2}\theta^{mj}E_mD_j\tilde{\psi} = 0 \tag{4.58}$$

$$iD_0{}'\tilde{\psi}' + \frac{1}{2\tilde{m}}D_i{}'D_i{}'\tilde{\psi}' + \frac{i}{2}\theta^{mj}E_m{}'D_j'\tilde{\psi}' = 0 \tag{4.59}$$

must hold in unprimed and primed frames respectively. Now making use of eq.(s) (4.55,4.56,4.57) in eq.(4.59) and then using eq.(4.58) we get the following condition involving η :

$$D_1\tilde{\psi} + i\partial_0\eta\tilde{\psi} = \left[-\frac{1}{\tilde{m}}\partial_j\eta - \frac{\theta}{2}\epsilon^{ij}F_{i1}\right]D_j\tilde{\psi} + \left[-\frac{1}{2\tilde{m}}\nabla^2\eta - \frac{\theta}{2}\epsilon^{ij}E_i\partial_j\eta\right]\tilde{\psi} \ . \tag{4.60}$$

Since we have considered the boost along the x-axis, the variable η occuring in the phase factor in eq.(4.56) will not have any x^2 dependence. Consequently we can set $\partial_2\eta = 0$. Also since we have taken the background electric field E_i to be constant, we have to consider here two independent possibilities : E along the direction of the boost and E perpendicular to the direction of the boost. Let us consider the former possibility first. Clearly in this case the term $\epsilon^{ij}E_i\partial_j\eta$ in the right hand side of eq.(4.60) vanishes and the above equation becomes

$$D_1\tilde{\psi} + i\left(\partial_0\eta\right)\tilde{\psi} = \left[-\frac{1}{\tilde{m}}\partial_1\eta - \frac{\theta B}{2}\right]D_1\tilde{\psi} - \frac{1}{2\tilde{m}}\left(\partial_1^2\eta\right)\tilde{\psi} \ . \tag{4.61}$$

Equating the coefficients of $D_1\tilde{\psi}$ and ψ from both sides we get the following conditions on η.

$$\left[\frac{1}{\tilde{m}}\partial_1\eta + \frac{\theta B}{2}\right] = -1 \tag{4.62}$$

$$i\partial_0\eta = -\frac{1}{2\tilde{m}}\partial_1^2\eta \ . \tag{4.63}$$

It is now quite trivial to obtain the following time-independent ($\partial_0\eta = 0$) real solution for η :

$$\eta = -\tilde{m}\left(1 + \frac{\theta B}{2}\right)x^1 \ . \tag{4.64}$$

This shows that boost in the direction of the electric field is a symmetry for the system. This is, however, not true when electric field is perpendicular to the direction of the boost. This can be easily seen by re-running the above analysis for this case, when instead of eq.(4.63) one gets

$$i\partial_0\eta = -\frac{1}{2\tilde{m}}\partial_1^2\eta + \frac{\theta E}{2}\partial_1\eta \tag{4.65}$$

along with eq.(4.62) which, however, remains unchanged. Clearly this pair (eq.(s) (4.62, 4.65)) does not admit any real solution. In fact, the solution can just be read off as

$$\eta = -\tilde{m}\left(1 + \frac{\theta B}{2}\right)x^1 + \frac{i}{2}\theta E\tilde{m}t \ . \tag{4.66}$$

This complex solution of η implies the wave-function (4.56) does not preserve its norm under this boost transformation as this transformation is no longer unitary. This demonstrates that the boost in the perpendicular direction of the applied electric field is not a symmetry of the system. This is clearly a noncommutative effect as it involves the noncommutative parameter θ. This violation of boost symmetry rules out the possibility of Galilean symmetry, let alone any exotic Galilean symmetry obtained by [30] in their model. We shall however see in chapter 7 that a twisted version of Galileo group can be made compatible with noncommutativity. Indeed, it turns out that the Galilean boost symmetry is taken care of rather trivially there, despite the appearance of mass as a central charge in Galilean algebra.

4.5 Hall Conductivity in commutative variables

In this section, we are going to compute the effect of noncommutativity on Hall conductivity, if any, using the formalism we have developed in section 4.3, in particular eq.(s) (4.35), (4.36). The violation of Galilean boost symmetry observed in the preceding section is not expected to interfere with this computation. Admittedly, the value of θ is very small, if it has its origin in the fundamental noncommutativity of nature, if any. On the other hand, the presence of electric field is known to lift the degeneracy of the Landau level, but the basic noncommutativity of the coordinates of the particle confined in the first Landau level (given in terms of the reciprocal of the magnetic field) is expected to persist even in the presence of a very small electric field. If this is really true then the value of the noncommutative parameter may be appreciable even for any condensed matter experiment that one can think of. We are however not going to discuss about this issue any further. The sole objective of the following exercise is to just illuminate the formalism we have developed so far.

Hence we now take up the problem of Hall effect in terms of commutative variables and attempt to solve the equation of motion (4.35) in Landau gauge.

$$A_0 = Ex^1, A_1 = 0, A_2 = Bx^1 \ . \tag{4.67}$$

Taking the trial solution of standard Landau gauge problem, appropriate for the gauge fixing condition (4.67)

$$\tilde{\psi}(t, x^1, x^2) = e^{-i\omega t} e^{ip_2 x^2} \phi(x^1) \tag{4.68}$$

we obtain

$$\left[\omega + Ex^1 - \frac{1}{2\tilde{m}}\left\{-\partial_1^2 + \left(p_2 - Bx^1\right)^2\right\} - \frac{\theta E}{2}\left(p_2 - Bx^1\right)\right]\phi(x^1) = 0. \tag{4.69}$$

Now using the following change of variables

$$x^1 \to X = x^1 - \frac{p_2 + \tilde{m}E/B}{B} \tag{4.70}$$

we get the equation

$$\left[-\frac{1}{2\tilde{m}}\partial_X^2 + \frac{\tilde{m}\tilde{\omega}_c^2}{2}\left(X - \frac{\tilde{m}E\theta}{2B}\right)^2\right]\phi^{'}(X) = \xi\phi^{'}(X) \tag{4.71}$$

where, $\phi^{'}(X) = \phi(x^1)$ and

$$\xi \;=\; \left[\left(\omega + p_2 E/B + \frac{\tilde{m}}{2}\left(E/B\right)^2\right) + \frac{\tilde{m}}{2}\theta\left(E^2/B\right)\right]. \tag{4.72}$$

A further change of variables

$$\bar{X} = (X - \frac{\tilde{m}E\theta}{2B}) \tag{4.73}$$

yields the standard harmonic oscillator equation with an enhanced frequency $\tilde{\omega}_c = (1 + \theta B)\omega_c$:

$$\left[-\frac{1}{2\tilde{m}}\partial_{\bar{X}}^2 + \frac{\tilde{m}\tilde{\omega}_c^2}{2}\bar{X}^2\right]\phi^{''}\left(\bar{X}\right) = \xi\phi^{''}\left(\bar{X}\right) \tag{4.74}$$

where, $\phi^{''}(\bar{X}) = \phi^{'}(X) = \phi(x^1)$ and ξ is the harmonic oscillator energy eigen-value. The admissible eigen-functions are given in terms of Hermite polynomials as

$$\phi_n^{''}(\bar{X}) = C_n \exp(-\frac{\tilde{m}\tilde{\omega}_c}{2}\bar{X}^2) H_n\left(\sqrt{\tilde{m}\tilde{\omega}_c}\bar{X}\right) \tag{4.75}$$

and the eigen-values are

$$\xi_n = (n + \frac{1}{2})\tilde{\omega}_c. \tag{4.76}$$

Also note that the θ-dependent term appearing in the harmonic oscillator energy eigen-value ξ (4.72) is due to electric field term in eq.(4.34). This will imply a quantization condition for ω

$$\omega_n = (n + \frac{1}{2})\tilde{\omega}_c - \left[\left(p_2 E/B + \frac{\tilde{m}}{2}\left(E/B\right)^2\right) + \frac{\tilde{m}}{2}\theta\left(E^2/B\right)\right]. \tag{4.77}$$

This indicates that the degeneracy of the Landau level has now been lifted by the external electric field as states with different p_2 values will have different energy eigen-values ω_n. Now the normalisation condition (4.33) becomes

$$1 = \int d\bar{X} dx^2 |\phi^{''}(\bar{X})|^2 \tag{4.78}$$

which for a sample width L_y yields the condition

$$\int d\bar{X} |\phi^{''}(\bar{X})|^2 = \frac{1}{L_y} \ . \tag{4.79}$$

Since $\hat{j}_1 = 0$, corresponding to the wave-function (4.68), the longitudinal current vanishes. Now coming to the transverse current, we note that $I_2^{(1)}$ contains only x^1 integration. This indicates that only integration by parts over x^1 variable can be performed, so that one can write, for example $\int dx^1 \tilde{\psi}^\dagger \left(D_1\tilde{\psi}\right) = -\int dx^1 \left(D_1\tilde{\psi}\right)^\dagger \tilde{\psi}$. However such an expression with $D_1 \to D_2$ in the above equation can also be written for the *particular form* of the wave-function (4.68) we have chosen to work with. In fact, this equality holds between the integrands themselves as one gets

$$\tilde{\psi}^\dagger D_2 \tilde{\psi} = -(D_2\tilde{\psi})^\dagger \tilde{\psi} = i(p_2 - A_2)(\phi(x^1))^2 \ . \tag{4.80}$$

One can therefore write $\int dx^1 \tilde{\psi}^\dagger \left(D_2\tilde{\psi}\right) = -\int dx^1 \left(D_2\tilde{\psi}\right)^\dagger \tilde{\psi}$ and the same thing also holds for higher order covariant derivatives appearing in $\hat{j}_2$, as one can verify. We can therefore write $I_2^{(1)}$ more compactly as

$$I_2^{(1)} = \int dx^1 \frac{1}{2\tilde{m}i}\left(1 - \frac{\theta B}{2}\right)\left\{\tilde{\psi}^\dagger\left(D_2\tilde{\psi}\right) - \left(D_2\tilde{\psi}\right)^\dagger \tilde{\psi}\right\}. \tag{4.81}$$

Here the $\left(1 - \frac{\theta B}{2}\right)$-factor[8] stems from the presence of the electric field term in the action (4.34). Now using eq.(4.80), the pair of co-ordinate transformations (4.70, 4.73) and eq.(4.79) one can

[8]As we are expressing everything in terms of the renormalised mass (i.e the observed mass) so there is no point in absorbing the factor $(1 - \theta B)$ in $\tilde{m}$ to give m.

cast the expression for transverse current for a single particle as

$$I_2^{(1)} = -\int d\bar{X} E \left(\frac{1}{B} + \frac{\theta}{2}\right)\left(1 - \frac{\theta B}{2}\right) |\phi^{''}(\bar{X})|^2 = -\frac{1}{L_y}\left(\frac{E}{B}\right). \tag{4.82}$$

Observe that $I_2^{(1)}$ is independent of both the indices n and p_2, so that all the electronic states $|n, p_2\rangle$ carry the same Hall current just as happens usually. Therefore to obtain the total current I_2, (following [85]) we just multiply $I_2^{(1)}$ (eq.(4.82)) by the number of available states ($\rho L_x L_y$) within an arbitrarily chosen rectangular area $L_x L_y$, where ρ is the density of such states. We therefore have the total current (upto order θ) as

$$I = -\rho L_x \frac{E}{B} = -\frac{\rho}{B} V \tag{4.83}$$

where, $V = E L_x$. Hence the Hall-conductivity $\sigma_H = \frac{I_2}{V} = -\frac{\rho}{B}$ has no explicit θ-dependence. At this stage one can easily see that the usual expression for degeneracy per unit area (for $E = 0$) holds, enabling one to define the filling fraction[9] in the conventional way, using which one can write down an alternative expression for Hall conductivity as $\sigma_H = -\frac{\nu}{2\pi}$.

4.6 Summary

In this chapter we have obtained an effective $U(1)$ gauge invariant action and correspondingly $U(1)$ gauge covariant Schrödinger equation starting from $U(1)_\star$ gauge invariant action, describing noncommutative Schrödinger field coupled to a background noncommutative $U(1)_\star$ gauge field, by using SW map followed by wave-function and mass renormalisation. The effect of noncommutativity on the mass parameter appears naturally in our analysis. Interestingly, we observe that the external magnetic field has to be static and uniform in order to get a canonical

[9]Note that the filling fraction ν can be defined because the expression of $\hat{j}_0$ in eq.(4.43) suggests (using eq.(s) (4.68),(4.70),(4.73)) that the centre of the harmonic oscillator, i.e. the centre of the charge distribution now will be located at $x^1 = p_2/(B)$. Now a range $\Delta x^1 = L_x$ for x^1 implies a range $\Delta p_2 = B L_x$ for p_2 which clearly can accomodate $\frac{\Delta p_2}{2\pi/L_y} = \frac{B}{2\pi} L_x L_y$ number of charged states within an area $L_x L_y$, if periodic boundary condition is imposed in the x^2-direction. One thus recovers the usual expression for degeneracy per unit area to be $B/(2\pi)$ with no accompanying noncommutative corrections.

form of Schrödinger equation upto θ-corrected terms, so that a natural probabilistic interpretation emerges. The Galilean symmetry of the model is next investigated where the translation and the rotation generators are seen to form a closed Euclidean sub-algebra of Galilean algebra. However, the boost is not found to be a symmetry of the system. This shows that even though the condition $\theta^{0i} = 0$ is Galilean invariant, a violation in the Galilean symmetry is exhibited for boost perpendicular to the electric field. Further, as a quantum mechanical application of our model, we take up the problem of Hall effect, where we compute the Hall conductivity (considering a set of free particles) and find no noncommutative correction upto first order in θ. Thus, in our formalism we reproduce the standard result of Hall conductivity with the filling fraction ν taking all possible values. The presence of impurities/disorder are essential for any quantization of ν, appropriate for Quantum Hall effect (integer/fractional), which may be the topic of future investigation.

Chapter 5

Dual families of noncommutative quantum systems

We have seen in the previous chapter that the SW map provides a correspondence from the noncommutative to the commutative space which preserves the gauge invariance and the physics [14]. However, it should be noted that this map is classical in nature, and therefore it is not clear whether this map will hold at the quantum level or not [86], [87], [88], [89]. It is therefore natural to enquire about the status of this map in noncommutative quantum mechanics where, apart from a few works [15, 83] which consider the SW map only to lowest order in the noncommutative parameter, very little has been done.

A second motivation for the present work comes from the by now well known noncommutative paradigm associated with the quantum Hall effect [65, 29, 31]. In particular, [80] explores the possibility of tuning the noncommutative parameter θ such that the electrons moving in two dimensional noncommutative space (in presence of both uniform external magnetic and electric fields) can be interpreted as either leading to the fractional quantum Hall effect or composite fermions in the usual coordinates. On the other hand, the discovery of the fractional quantum Hall effect led to the immediate realization that the Coulomb interaction plays an essential role in the understanding of this phenomenon [90]. This raises the question whether the noncommutative Hamiltonian introduced by [80] in a somewhat ad hoc way can be reinterpreted as an effective noncommutative Hamiltonian which describes the same physics as the interacting commutative theory, at least in some approximation. Clearly, this equivalence cannot be exact as it is well known [44, 15] that a noninteracting commutative Hamiltonian with constant magnetic field maps onto a noninteracting noncommutative Hamiltonian with constant magnetic field. However, one might think about the possibility that there is some preferred value of the noncommutative parameter which minimizes the interaction on the noncommutative level. If this is the case the corresponding noninteracting noncommutative Hamiltonian might be a good starting point for a computation which treats the residual interaction as a perturbation. This might seem problematic due to the degeneracy of the Landau levels. However, under the assumption of a central potential this construction can be carried out in each angular momentum sector, which effectively lifts this degeneracy and allows for a perturbative treatment in each sector (see section 5.5).

With the above remarks in mind, i.e. the physical equivalence of different noncommutative descriptions, the following question arises quite naturally: how should a family of noncommutative Hamiltonians be parameterized as a function of the noncommutative parameter to ensure that they are physically equivalent? This is the central issue addressed here. The relation to the SW map and the possible use to construct dualities are natural secondary issues that arise which has also been addressed here, although not in complete generality.

This chapter is organized as follows. In section 5.1, the general construction of a one parameter family of noncommutative, physically equivalent Hamiltonians is considered. In section 5.2 and 5.3, application of this general construction is carried out to a particle in two dimensions moving in a constant magnetic field without interactions and in the presence of a harmonic potential, respectively. The construction is carried out to all orders in the noncommutative parameter. The relation between this construction and the SW map is discussed in section 5.4. In section 5.5, an approximate duality between the interacting commutative Hamiltonian and a noninteracting noncommutative Hamiltonian is constructed for an harmonic oscillator potential. Section 5.6 contains our discussion and conclusions. An appendix summarizes notational issues at the end.

5.1 General considerations

We consider a NR particle moving in a plane under a potential V and coupled minimally to a $U(1)$ gauge field A. In commutative space the Hamiltonian reads ($\hbar = c = e = 1$)

$$H = \frac{(\mathbf{p} - \mathbf{A})^2}{2m} + V(x). \tag{5.1}$$

The prescription to go over to the noncommutative space is to replace the commutative quantities by noncommutative ones, denoted by a hat, and introduce the star product, defined in the usual way (4.9). The spacetime noncommutativity is assumed to vanish ($\theta^{0i} = 0$) and, for a planar system, the spatial part of the θ-matrix can be written as $\theta^{ij} = \theta\epsilon^{ij}$. The Schrödinger equation in noncommutative space therefore reads

$$\begin{aligned} i\frac{\partial\hat{\psi}(\mathbf{x},t)}{\partial t} &= \left[\frac{\left(\mathbf{p} - \hat{\mathbf{A}}\right) \star \left(\mathbf{p} - \hat{\mathbf{A}}\right)}{2\hat{m}} + \hat{V}(x)\right] \star \hat{\psi}(\mathbf{x},t) \\ &= \hat{H} \star \hat{\psi}(\mathbf{x},t) \equiv \hat{H}_{BS}(\theta)\hat{\psi}(\theta). \end{aligned} \tag{5.2}$$

Here, $\hat{H}_{BS}(\theta)$ denotes the Hamiltonian after the star product has been replaced by a Bopp-shift, defined by [65, 43, 44]

$$\left(\hat{f} \star \hat{g}\right)(x) = \hat{f}\left(x - \frac{\theta}{2}\epsilon^{ij}p_j\right)\hat{g}(x). \tag{5.3}$$

Note that the quantities appearing in $\hat{H}_{BS}(\theta)$ are still the noncommutative ones.

The condition that the physics remains invariant under a change in θ requires that $\hat{H}_{BS}(\theta)$ and $\hat{H}_{BS}(0)$ are related by a unitary transformation

$$\hat{H}_{BS}(\theta) = U(\theta)\hat{H}_{BS}(0)U^{\dagger}(\theta) \tag{5.4}$$

and that

$$\hat{\psi}(\theta) = U(\theta)\hat{\psi}(0)\ . \tag{5.5}$$

Differentiating eq.(5.4) with respect to θ, we obtain

$$\frac{d\hat{H}_{BS}(\theta)}{d\theta} = [\eta(\theta), \hat{H}_{BS}(\theta)] \tag{5.6}$$

where,

$$\eta(\theta) = \frac{dU(\theta)}{d\theta} U^\dagger(\theta) \tag{5.7}$$

is the generator of the unitary transformation relating the noncommutative Bopp-shifted Hamiltonian with the commutative Hamiltonian.

We now consider under what conditions eq.(5.6) admits a solution for η. These conditions will, of course, provide us with the constraints on the parameterization of the noncommutative Hamiltonian necessary to ensure unitary equivalence, i.e., the existence of η. It is a simple matter to verify that eq.(5.6) admits a solution for η if and only if

$$\langle n, \theta| \frac{d\hat{H}_{BS}(\theta)}{d\theta} |n, \theta\rangle = 0 \quad , \quad \forall n \tag{5.8}$$

where, $|n, \theta\rangle$ are eigenstates of $\hat{H}_{BS}(\theta)$, i.e.,

$$\hat{H}_{BS}(\theta)|n, \theta\rangle = E_n |n, \theta\rangle. \tag{5.9}$$

If eq.(5.8) holds, the off-diagonal part of η is uniquely determined by

$$\eta = \sum_{n \neq m} \frac{\langle n, \theta| \frac{d\hat{H}_{BS}}{d\theta} |m, \theta\rangle}{E_m - E_n} |n, \theta\rangle\langle m, \theta| \tag{5.10}$$

while the diagonal part is arbitrary, reflecting the arbitrariness in the phase of the eigenstates. Here we have assumed no degeneracy in the spectrum of $\hat{H}_{BS}(\theta)$. The generalization to the case of degeneracies is straightforward.

The set of conditions (5.8) should be viewed as the set of conditions which determines the θ-dependency of the matrix elements of the noncommutative potential $\hat{V}$ and gauge field $\hat{A}$. Expectedly these matrix elements are under-determined, i.e., that not both $\hat{V}$ and $\hat{A}$ are uniquely determined by them. Instead one can fix one of these and compute the other. For comparison with the SW map, it is therefore natural to take for $\hat{A}$ the noncommutative gauge field as determined from the SW map. Note that this procedure implies that $\hat{V}$ will be gauge dependent.

Consider the SW map for the noncommutative wave-function (4.20). Below we consider two dimensional systems in a constant magnetic field. Taking the symmetric gauge, the SW map reduces to a θ dependent scaling transformation. Clearly this is not a unitary transformation and a unitary SW map can be constructed as in [83]. However, a more convenient point of view, closer in spirit to the SW map, would be to relax the condition of unitarity above. It therefore seems worthwhile, in particular to relate to the SW map, to generalize the above considerations by relaxing the condition of unitarity.

This generalization is straightforward. The unitary transformation in eq.(s) (5.4) and (5.5) needs to be replaced by a general similarity transformation

$$\hat{H}_{BS}(\theta) = S(\theta)\hat{H}_{BS}(0)S^{-1}(\theta) \tag{5.11}$$

while

$$\hat{\psi}(\theta) = S(\theta)\hat{\psi}(0) \tag{5.12}$$

and note that a new inner product $\langle\psi|\phi\rangle_T = \langle\psi|T|\phi\rangle$ can be defined such that $\hat{H}_{BS}(\theta)$ is hermitian with respect to it. In particular T is given by $T = (S^{-1})^\dagger S^{-1}$ and has the property $T\hat{H}_{BS}(\theta) = \hat{H}^\dagger_{BS}(\theta)T$. Under this prescription the same physics results. A detailed exposition of these issues can be found in [91].

Differentiating eq.(5.11) with respect to θ, we obtain

$$\frac{d\hat{H}_{BS}(\theta)}{d\theta} = [\eta(\theta), \hat{H}_{BS}(\theta)] \tag{5.13}$$

where,

$$\eta(\theta) = \frac{dS(\theta)}{d\theta}S^{-1}(\theta) \tag{5.14}$$

is now the generator of the similarity transformation relating the noncommutative Bopp-shifted Hamiltonian with the commutative Hamiltonian.

It can now be easily verified that eq.(5.8) gets replaced by

$$\langle n, \theta|T\frac{d\hat{H}_{BS}(\theta)}{d\theta}|n, \theta\rangle = 0 \quad , \quad \forall n \tag{5.15}$$

where, $|n, \theta\rangle$ are eigenstates of $\hat{H}_{BS}(\theta)$ (note that the eigenvalues will be real as $\hat{H}_{BS}(0)$ is assumed to be hermitian and thus has real eigenvalues). As before, if eq.(5.15) holds, the off-diagonal part of η is uniquely determined by

$$\eta = \sum_{n \neq m} \frac{\langle n, \theta | T \frac{d\hat{H}_{BS}}{d\theta} | m, \theta \rangle}{E_m - E_n} |n, \theta\rangle \langle m, \theta | T \tag{5.16}$$

while the diagonal part is arbitrary, reflecting the arbitrariness in the phase and now also the normalization of the eigenstates.

Under the above description, the Hamiltonians $\hat{H}_{BS}(\theta)$ and $\hat{H}_{BS}(0)$ are physically equivalent. There is, however, one situation in which this equivalence may break down and of which careful note should be taken. This happens when the similarity transformation $S(\theta)$ becomes singular for some value of θ, which will be reflected in the appearance of zero norm or unnormalizable states in the new inner product. Only values of θ which can be reached by integrating eq.(5.14) from $\theta = 0$ without passing through a singularity, can be considered physically equivalent to the commutative system.

To solve eq.(s) (5.8) or (5.15) in general is of course impossible. Therefore we take a slightly different approach in what follows. An ansatz for η motivated by the SW map is taken to solve eq.(5.6) or eq.(5.13) directly. We have already noted above that in the cases of interest to us, i.e., two dimensional systems in constant magnetic fields, the SW map for the noncommutative wave-function corresponds to a scaling transformation in the symmetric gauge. This motivates us to make the following ansatz

$$\eta(\theta) = f(\theta) r \partial_r = i f(\theta) x.p \tag{5.17}$$

with f being an arbitrary function to be determined. The finite form of this scaling transformation can be readily obtained by integrating eq.(5.14) to yield

$$S(\theta) = e^{i\left(\int_0^\theta f(\theta')d\theta'\right)x.p}. \tag{5.18}$$

Clearly this is not a unitary transformation and therefore falls in the class of more general transformations described above eq.(5.11). Furthermore we note that the non-singularity of $S(\theta)$ requires that the integral $\int_0^\theta f(\theta')d\theta'$ exists.

5.2 Free particle in a constant magnetic field

In this section, we apply the considerations discussed above to the case of a free particle ($\hat{V} = 0$) moving in a noncommutative plane in the presence of a constant noncommutative magnetic field. The Schrödinger equation is given by eq.(5.2) with $\hat{V}$ set to zero.

In the symmetric gauge $\hat{A}_i = -\frac{\bar{B}(\theta)}{2}\epsilon_{ij}x_j$ [1], the Bopp-shifted Hamiltonian (5.2) is easily found to be

$$\begin{aligned}\hat{H}_{BS}(\theta) &= \frac{\left(1+\frac{\bar{B}\theta}{4}\right)^2}{2\hat{m}(\theta)}\left(\mathbf{p} - \frac{1}{1+\frac{\bar{B}\theta}{4}}\mathbf{A}\right)^2 \\ &= \frac{1}{2M(\theta)}\left(p_x^2 + p_y^2\right) + \frac{1}{2}M(\theta)\Omega(\theta)^2\left(x^2+y^2\right) \\ &\quad -\Omega(\theta)L_z \end{aligned} \tag{5.19}$$

where,

$$\frac{1}{2M(\theta)} = \frac{\left(1+\frac{\bar{B}\theta}{4}\right)^2}{2\hat{m}(\theta)} \quad , \quad \frac{1}{2}M(\theta)\Omega(\theta)^2 = \frac{\bar{B}^2}{8\hat{m}(\theta)} \ . \tag{5.20}$$

Substitution of the above form of the Hamiltonian in eq.(5.6) with η as in eq.(5.17), leads to the following set of differential equations :

$$\frac{dM^{-1}(\theta)}{d\theta} = -2f(\theta)M^{-1}(\theta) \tag{5.21}$$

$$\frac{d\left(M(\theta)\Omega(\theta)^2\right)}{d\theta} = 2M(\theta)\Omega(\theta)^2 f(\theta) \tag{5.22}$$

$$\frac{d\Omega(\theta)}{d\theta} = 0. \tag{5.23}$$

Eq.(5.23) ensures the stability of the energy spectrum, i.e the cyclotron frequency $\Omega(\theta) = \Omega(\theta = 0) = B/2m$, where $m = \hat{m}(\theta = 0)$. This is the physical input in our analysis and will play a very important role as we shall see later. The above equations (5.20, 5.22, 5.23) immediately

[1] We use $\bar{B}(\theta)$ to denote the noncommutative counterpart of B in eq.(5.25). It should not be confused with the noncommutative magnetic field $\hat{B}$ as determined from the field strength (see eq.(5.28)). In the limit $\theta = 0$, $\bar{B}(\theta) = B$.

lead to

$$f(\theta) = \frac{1}{2M(\theta)}\frac{dM(\theta)}{d\theta} = \frac{\partial_\theta \bar{B}(\theta) - \frac{\bar{B}(\theta)^2}{4}}{2\bar{B}(\theta)\left(1 + \frac{\theta\bar{B}(\theta)}{4}\right)} \tag{5.24}$$

which fixes f once $\bar{B}$ has been determined. As indicated before, we take $\hat{A}$ as the noncommutative gauge field determined from the SW map. With this in mind we now proceed to determine $\bar{B}$.

It is easy to see that a symmetric gauge configuration

$$A_i = -\frac{B}{2}\epsilon_{ij}x^j \tag{5.25}$$

with magnetic field $B = F_{12} = (\partial_1 A_2 - \partial_2 A_1)$, transforms to a symmetric gauge field configuration at the noncommutative level under the SW transformation (4.21). Using the same notation as in eq.(5.25), we write

$$\hat{A}_i = -\frac{\bar{B}}{2}\epsilon_{ij}x^j \tag{5.26}$$

where, $\bar{B}$ is determined to leading order in θ from eq.(4.21) to be

$$\bar{B} = B\left(1 + \frac{3\theta B}{4}\right). \tag{5.27}$$

Note that $\bar{B}(\theta)$ should not be identified with the noncommutative magnetic field $\hat{B}$, which has an additional Moyal bracket term $[\hat{A}_1, \hat{A}_2]_\star$:

$$\hat{B} = \hat{F}_{12} = \partial_1\hat{A}_2 - \partial_2\hat{A}_1 - i(\hat{A}_1\star\hat{A}_2 - \hat{A}_2\star\hat{A}_1) = \bar{B}(1 + \frac{\theta\bar{B}}{4}). \tag{5.28}$$

This is precisely the same expression one gets if one applies the SW map directly at the level of the field strength tensor, which is given by [59]:

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + \theta\epsilon^{ij}F_{\mu i}F_{\nu j}\ . \tag{5.29}$$

Note that the expression (5.28) relating $\hat{B}$ with $\bar{B}$ is an exact one in contrast with eq.(5.27) which relates $\bar{B}$ to B only up to leading order in θ. For a constant field configuration, the SW equation for the field strength tensor can be integrated exactly to give the result [59]

$$\hat{B} = \frac{1}{1-\theta B}B. \tag{5.30}$$

From eq.(s) (5.28) and (5.30), we obtain a quadratic equation in $\bar{B}(\theta)$ that can be solved exactly to give

$$\bar{B}(\theta) = \frac{2}{\theta}\left[(1-\theta B)^{-1/2} - 1\right] . \tag{5.31}$$

The above expression for $\bar{B}(\theta)$ is exact up to all orders in θ. When substituted in eq.(5.26) an expression, correct to all orders in θ, for the noncommutative gauge field $\hat{A}_i$ result

$$\hat{A}_i = -\frac{1}{\theta}\left[(1-\theta B)^{-1/2} - 1\right]\epsilon_{ij}x^j . \tag{5.32}$$

Substituting $\bar{B}(\theta)$ from eq.(5.31) into eq.(5.24) yields

$$f(\theta) = \frac{\bar{B}(\theta)}{4} . \tag{5.33}$$

Upon differentiating eq.(5.12) with respect to θ and using f from eq.(5.33), we find that $\hat{\psi}(\theta)$ must satisfy the following equation:

$$\frac{d\hat{\psi}(\theta)}{d\theta} = \frac{\bar{B}(\theta)}{4} r \frac{d\hat{\psi}(\theta)}{dr} . \tag{5.34}$$

This result can now be compared to the corresponding SW transformation rule for $\hat{\psi}$. The SW equation (4.20) for an arbitrary $\theta + \delta\theta$ reads

$$\hat{\psi}(\theta+\delta\theta) - \hat{\psi}(\theta) = -\frac{1}{2}\theta\epsilon^{ij}\hat{A}_i \star \partial_j\hat{\psi}(\theta). \tag{5.35}$$

Upon substituting $\hat{A}_i$ from eq.(5.26), eq.(5.34) indeed results. Thus the transformation rule as obtained from the requirement of physical equivalence agrees with that of the SW map.

Finally, substituting $\bar{B}(\theta)$ in the condition $\Omega = B/2m$ yields the following expression for $\hat{m}(\theta)$:

$$\hat{m}(\theta) = \frac{m}{1-\theta B} . \tag{5.36}$$

The above equation relates the noncommutative mass $\hat{m}(\theta)$ with the commutative mass m. This generalizes the result obtained in eq.(4.30) to all orders in θ.

The Schrödinger equation can of course be solved exactly in a simple case such as this. It is useful to see what the above procedure entails from this point of view. To solve for the eigenvalues and eigenfunctions of eq.(5.19) is a standard procedure and for notational completeness we summarize the essential steps in appendix. This results in the degenerate eigenvalue spectrum

$$E_{n_-,\ell} = 2\Omega\left(n_- + \frac{1}{2}\right)$$
$$n_- = 0, 1, \ldots \quad ; \; \ell = -n_-, -n_- + 1 \ldots \tag{5.37}$$

where, ℓ denotes the eigenvalues of the angular momentum operator L_3. The corresponding eigenstates are obtained by acting with the creation operators $b_\pm^\dagger$ defined in eq.(5.81) on the ground-state

$$\begin{aligned} \hat{\psi}(z, \bar{z}; \theta) &= N \exp\left[-\frac{M\Omega}{2}\bar{z}z\right] \\ &= N \exp\left[-\frac{\bar{B}(\theta)}{4\left(1 + \frac{\bar{B}(\theta)\theta}{4}\right)}\bar{z}z\right] \; . \end{aligned} \tag{5.38}$$

Comparing with our previous results, we note that eq.(5.23) ensures invariance of the spectrum under a change of θ. Furthermore direct inspection shows that the *unnormalised* ground-state and, subsequently, also all excited states satisfy the transformation rule (5.34). The fact that the unnormalised wave-functions satisfy the transformation rule (5.34) is consistent with our earlier remarks on the non-unitary nature of the scaling transformation.

Finally, note that although the noncommutative parameters $\bar{B}(\theta)$ and $\hat{m}(\theta)$ have singularities at $\theta = 1/B$ [2], these singularities cancel in the parameter Ω, which is by construction free of any singularities, i.e., the spectrum is not affected by this singularity. This is also reflected by the fact that the integral of f, as determined in eq.(5.33), is free of this singularity. Thus, despite the appearance of this singularity in the parameters of the noncommutative Hamiltonian, there is no breakdown of the physical equivalence (see the discusion in section 5.1).

5.3 Harmonic oscillator in a constant magnetic field

In this section, we include a harmonic oscillator potential $V = \lambda r^2$ in the commutative Hamiltonian (5.1). If the physical equivalence between the noncommutative and commutative Hamiltonians is indeed implementable through a scale transformation, we expect the potential to be

[2] This singularity was also encountered in the previous chapter.

form preserving (this is certainly not true for arbitrary potentials). We therefore take for the noncommutative potential in eq.(5.2) $\hat{V} = \hat{\lambda}(\theta) r^2$, where the oscillator strength $\hat{\lambda}(\theta)$ has to be determined. Obviously we must also require that $\hat{\lambda}(\theta) = \lambda$ in the limit $\theta = 0$. The Bopp-shifted Hamiltonian with this form for the noncommutative Hamiltonian (5.2), is easily found to be

$$\begin{aligned} \hat{H}_{BS}(\theta) &= \frac{\left(1+\frac{\bar{B}\theta}{4}\right)^2}{2\hat{m}} \left(\mathbf{p} - \frac{1}{1+\frac{\bar{B}\theta}{4}}\mathbf{A}\right)^2 \\ &\quad + \hat{\lambda}(\theta)\left[\frac{\theta^2}{4}\left(p_x^2+p_y^2\right) + \left(x^2+y^2\right) - \theta L_z\right] \\ &= \frac{1}{2M}\left(p_x^2+p_y^2\right) + \frac{1}{2}M\Omega^2\left(x^2+y^2\right) \\ &\quad - \Lambda(\theta) L_z \end{aligned} \tag{5.39}$$

where,

$$\begin{aligned} \frac{1}{2M} &= \frac{\left(1+\frac{\bar{B}\theta}{4}\right)^2}{2\hat{m}} + \frac{\hat{\lambda}\theta^2}{4} \\ \frac{1}{2}M\Omega^2 &= \frac{\bar{B}(\theta)^2}{8\hat{m}(\theta)} + \hat{\lambda}(\theta) \\ \Lambda(\theta) &= \left[\frac{M\Omega^2\theta}{2} + \frac{\bar{B}\left[1-\left(\frac{M\Omega\theta}{2}\right)^2\right]}{2\left(1+\frac{\bar{B}\theta}{2}\right)M}\right] . \end{aligned} \tag{5.40}$$

Here $\bar{B}(\theta)$ is again taken from the SW map (5.31). Substituting the above form of the Hamiltonian in eq.(5.6) with η as in eq.(5.17), we obtain the following set of differential equations:

$$\frac{dM^{-1}(\theta)}{d\theta} = -2f(\theta)M^{-1}(\theta) \tag{5.41}$$

$$\frac{d\left(M(\theta)\Omega(\theta)^2\right)}{d\theta} = 2M(\theta)\Omega(\theta)^2 f(\theta) \tag{5.42}$$

$$\frac{d\Lambda(\theta)}{d\theta} = 0 . \tag{5.43}$$

Eq.(5.43) requires that $\Lambda(\theta)$ is independent of θ and hence we have the condition $\Lambda(\theta) = \Lambda(0) = B/2m$. Substituting the form of $M(\theta)$ in terms of $\hat{m}(\theta)$ and $\hat{\lambda}(\theta)$, we obtain the following solution for $\hat{m}(\theta)$ in terms of $\hat{\lambda}(\theta)$:

$$\hat{m}(\theta) = \frac{m}{(1-\theta B)}\frac{B}{\left(B - 2m\theta\hat{\lambda}(\theta)\right)} . \tag{5.44}$$

The set of differential equations in (5.42) can also be combined to obtain

$$\frac{d\Omega^2}{d\theta} = 0 \ . \tag{5.45}$$

This shows that Ω is a constant and therefore we have

$$\Omega^2(\theta) = \Omega^2(\theta = 0) = \frac{B^2}{4m^2} + \frac{2\lambda}{m} \ . \tag{5.46}$$

Substituting $\Omega^2(\theta)$ in eq.(5.40) and using eq.(5.44), a quadratic equation for $\hat{\lambda}(\theta)$ is obtained:

$$\begin{aligned}\left[B^3 + 8(1-\theta B)mB\hat{\lambda}(\theta) - 16(1-\theta B)m^2\theta\hat{\lambda}(\theta)^2\right] \\ = B^3 + 8\lambda mB \ .\end{aligned} \tag{5.47}$$

The solution for $\hat{\lambda}(\theta)$ yields

$$\hat{\lambda}(\theta) = \frac{B}{4m\theta}\left[1 - \left(1 - \frac{8\lambda m\theta}{B(1-\theta B)}\right)^{\frac{1}{2}}\right] \tag{5.48}$$

where we have taken the negative sign before the square root since with this choice we have $\hat{\lambda}(\theta = 0) = \lambda$.

With the value of $\bar{B}(\theta)$ fixed from the SW map and $\hat{m}(\theta)$ and $\hat{\lambda}(\theta)$ determined as above, we can compute the value of $M(\theta)$ from eq.(5.40) and subsequently the value of $f(\theta)$ from eq.(5.42) as $f(\theta) = \frac{1}{2M(\theta)}\frac{dM(\theta)}{d\theta}$. The expression is a lenghthy one and we do not need to list it here. What is important to note, however, is that once $f(\theta)$ is fixed, the transformation rule satisfied by $\hat{\psi}(\theta)$ is determined from eq.(5.12) and that this transformation rule is not the same as the one derived from the SW map (5.34). In fact, it turns out that the transformation rule for $\hat{\lambda}(\theta)$ is also different from the SW map. We discuss these points in more detail in the next section.

As a consistency check, one can once again solve for the eigenvalues and eigenstates. The procedure is the same as in appendix and one finds for the eigenvalues

$$\begin{aligned}E_{n_-,\ell} = 2\Omega\left(n_- + \frac{1}{2}\right) + (\Omega - \Lambda)\ell \\ n_- = 0, 1, \ldots; \ \ell = -n_-, -n_- + 1, \ldots \ .\end{aligned} \tag{5.49}$$

From the above expression of the energy eigenvalues, it is easy to see that the degeneracy in ℓ has been lifted. However, in the limit $\lambda = 0$, the energy spectrum given by eq.(5.37) is recovered. The corresponding eigenstates are again obtained by acting with the creation operators $b_\pm^\dagger$ defined in eq.(5.81) on the ground-state

$$\hat{\psi}(z,\bar{z};\theta) = N \exp\left[-\frac{1}{4}\sqrt{\frac{2\bar{B}(\theta)^2 + 16\hat{\lambda}\hat{m}}{2\left(1+\frac{\theta\bar{B}(\theta)}{4}\right)^2 + \theta^2\hat{\lambda}\hat{m}}}\bar{z}z \right] . \tag{5.50}$$

Once again we note that eq.(s) (5.42) and (5.43) ensures invariance of the spectrum under a change in θ. Using the values of $\bar{B}(\theta)$, $\hat{m}(\theta)$ and $\hat{\lambda}(\theta)$as determined above, one finds that the *unnormalised* wave-functions indeed satisfy the transformation rule as determined by eq.(5.12) and not the SW transformation rule (5.34). Also, in the $\theta = 0$ limit, eq.(5.50) smoothly goes over to the standard commutative result

$$\hat{\psi}(z,\bar{z},\theta=0) = \psi(z,\bar{z}) = N \exp\left[-\frac{1}{4}\sqrt{B^2 + 8\lambda m}\bar{z}z \right] . \tag{5.51}$$

Finally we remark on the non-singularity of the scaling transformation $S(\theta)$. As already pointed out in section 5.1, this requires the existence of the integral of f, which in the present case is simply given by $\log(M(\theta)/m)/2$. This turns out to be free of singularities, although the noncommutative parameters again exhibit singularities at $\theta = 1/B$. As in the free case these singularities cancel in the parameters Ω and Λ which determine the physical spectrum.

5.4 Connection with Seiberg-Witten map

In this section, we are going to discuss the relationship of the flow equations for $\hat{m}(\theta)$ and $\hat{\lambda}(\theta)$ obtained from the stability analysis of the previous section to the flow equation obtained from the SW map. To that end, let us write down the $U(1)_\star$ gauge invariant action from which the $\star$ gauge covariant one-particle Schrödinger equation (5.2) follows as Euler-Lagrangian equation:

$$\hat{S} = \int d^3x \hat{\psi}^\dagger \star (i\hat{D}_0 + \frac{1}{2\hat{m}}\hat{D}_i \star \hat{D}_i + \hat{V}) \star \hat{\psi} . \tag{5.52}$$

The preservation of $U(1)_\star$ gauge invariance of the action requires that the potential $\hat{V}$ must transform adjointly under $\star$ gauge transformation

$$\hat{V}(x) \longrightarrow \hat{V}'(x) = \hat{U}(x) \star \hat{V}(x) \star \hat{U}^\dagger(x) \tag{5.53}$$

for $\hat{U}(x) \in U(1)_\star$. The reason for this is quite simple to see. If it were to remain invariant, this would have implied that the Moyal bracket between $\hat{V}$ and $\hat{U}$, $\forall\ \hat{U} \in U(1)_\star$ vanishes ($[\hat{V}, \hat{U}]_\star = 0$). Through Wigner-Weyl correspondence (2.23), this in turn implies that V_{op} commutes with U_{op} at the operator level: $[V_{\text{op}}, U_{\text{op}}] = 0\ \forall U_{\text{op}}$. Applying Schur's lemma, assuming that U_{op} acts irreducibly, this indicates V_{op} =constant. Clearly this does not have the desired property. Now the SW transformation property of $\hat{V}(x)$ can be easily obtained as

$$\hat{V}'(x) = \hat{V}(x) - \delta\theta\epsilon^{ij}\hat{A}_i \star \partial_j \hat{V}(x) \tag{5.54}$$

which relates the noncommutative potential $\hat{V}(x;\theta) \equiv \hat{V}(x)$ for noncommutative parameter θ to the corresponding noncommutative potential $\hat{V}(x;\theta+\delta\theta) \equiv \hat{V}'(x)$ for noncommutative parameter $(\theta + \delta\theta)$. For the noncommutative gauge potential (5.26), this leads to the following differential equation

$$\frac{d\hat{V}(\theta)}{d\theta} = \frac{\bar{B}(\theta)}{2} r \frac{d\hat{V}(\theta)}{dr} \tag{5.55}$$

which can be solved by the method of seperation of variables[3], i.e. by taking $\hat{V}(r,\theta) = V(r)\hat{\lambda}_{sw}(\theta)$. We also have the boundary condition $\hat{\lambda}_{sw}(\theta = 0) = \lambda$. Using this, eq.(5.55) simplifies to

$$\frac{2}{\bar{B}(\theta)\hat{\lambda}_{sw}(\theta)} \frac{d\hat{\lambda}_{sw}(\theta)}{d\theta} = \frac{r}{V(r)} \frac{dV(r)}{dr} = k(= \text{constant}). \tag{5.56}$$

Solving we get

$$V(r) \;=\; \lambda r^k$$

[3]Such a seperation of variables can be made as one can expect that a commutative central potential goes over to another central potential of the same form but of different coupling constant at the noncommutative level. With this only the coupling constant is subjected to SW flow.

$$
\begin{aligned}
\hat{\lambda}_{sw}(\theta) &= \lambda \exp\left[\frac{k}{2}\int_0^\theta d\theta' \bar{B}(\theta')\right] \\
&= \lambda \left(\frac{1+(1-\theta B)^{\frac{1}{2}}}{2}\right)^{-2k} .
\end{aligned}
\tag{5.57}
$$

For $k = 2$, we get the usual harmonic oscillator, i.e.

$$
\begin{aligned}
\hat{V}(r,\theta) &= \hat{\lambda}_{sw}(\theta) r^2 \\
&= \lambda \left(\frac{1+(1-\theta B)^{\frac{1}{2}}}{2}\right)^{-4} r^2 .
\end{aligned}
\tag{5.58}
$$

If we now demand as in the free case that eq.(5.50) satisfies eq.(5.34) then the solution of eq.(5.34) can also be found by taking the trial solution $\hat{\psi}(z,\bar{z};\theta) = N\exp\left(-\frac{\bar{z}z}{4}g(\theta)\right)$ subject to the boundary condition (5.51) at $\theta = 0$. This leads to the solution

$$
\hat{\psi}_{sw}(z,\bar{z};\theta) = N\exp\left[-\bar{z}z\frac{\sqrt{(B^2+8m\lambda)}}{\left((1-\theta B)^{\frac{1}{2}}+1\right)^2}\right] . \tag{5.59}
$$

Comparing eq.(s) (5.50) and (5.59), we get an algebraic equation

$$
\frac{4\left(B^2+8m\lambda\right)^{\frac{1}{2}}}{\left[(1-\theta B)^{\frac{1}{2}}+1\right]^2} = \left[\frac{2\bar{B}^2(\theta)+16\hat{\lambda}_{sw}\hat{m}_{sw}}{2\left(1+\frac{\theta\bar{B}(\theta)}{4}\right)^2+\theta^2\hat{\lambda}_{sw}\hat{m}_{sw}}\right]^{\frac{1}{2}} \tag{5.60}
$$

which leads to

$$
\begin{aligned}
&\hat{\lambda}_{sw}\hat{m}_{sw} = \\
&\frac{4m\lambda\left[1+(1-\theta B)^{\frac{1}{2}}\right]^2}{(1-\theta B)\left(\{(1-\theta B)^{\frac{1}{2}}+1\}^4-\theta^2\left(B^2+8m\lambda\right)\right)} .
\end{aligned}
\tag{5.61}
$$

Substituting the value of $\hat{\lambda}_{sw}(\theta)$ from eq.(5.57), we obtain the value of $\hat{m}_{sw}(\theta)$ as

$$
\hat{m}_{sw}(\theta) = \frac{m}{4(1-\theta B)}\frac{\left[1+(1-\theta B)^{\frac{1}{2}}\right]^6}{\left[\{(1-\theta B)^{\frac{1}{2}}+1\}^4-\theta^2\left(B^2+8m\lambda\right)\right]} . \tag{5.62}
$$

The flow structure of $\hat{\lambda}_{sw}$ (eq.(5.57)) and $\hat{m}_{sw}$ (eq.(5.62)) in θ shows that the SW-flow is different (in the presence of interactions) from the flows obtained in the previous section eq.(s) (5.44) and

(5.48) from the consideration of the stability of the spectrum, although the formal structure of the wave-functions $\hat{\psi}_{sw}$ (eq.(5.59)) and $\hat{\psi}$ (eq.(5.50)) are the same. Indeed, it can be checked easily and explicitly that the flow obtained here (eq.(s) (5.57) and (5.62)) from the SW map is not spectrum preserving, as is the case with the flow of the previous section. This indicates that these flows are not equivalent or related in some simple way.

We have already seen that in absence of interaction ($\hat{\lambda} = 0$) the noncommutative wave-function $\hat{\psi}_{sw}$ satisfies the SW map, subject to the boundary condition (5.83) at $\theta = 0$, when $\hat{\psi}_{sw}$ becomes identifiable with the commutative wave-function ψ. Also, unlike its noncommutative counterpart $\hat{\psi}$, the commutative wave-function ψ does not have a flow of its own in θ. However, the situation changes drastically in the presence of interactions. To see this more clearly, let us consider the Schrödinger equation

$$iD_0\psi = -\frac{1}{2m}D_iD_i\psi - \frac{i\theta}{2}\epsilon^{ij}F_{i0}D_j\psi \tag{5.63}$$

obtained from the $U(1)$ gauge invariant effective action in the presence of a background gauge field, derived in the previous chapter to leading order in the noncommutative parameter θ. Note that the temporal component A_0 of the background gauge field can be regarded as $(-V)$, where V is the potential since this background gauge field is time independent. Indeed the SW transformation property of both A_0 and V become identical, as can be seen from eq.(s) (5.54) and (4.21). This helps us to identify, again to leading order in θ, the corresponding Hamiltonian as

$$H = \frac{(\mathbf{p} - \mathbf{A})^2}{2m} + V - \frac{\theta}{2}\epsilon^{ij}\partial_i V\,(p_j - A_j)\,. \tag{5.64}$$

For a central potential $V(r)$, this simplifies in the symmetric gauge (5.25) to

$$H = \frac{(\mathbf{p} - \mathbf{A})^2}{2m} + V - \frac{\theta}{2r}\frac{\partial V}{\partial r}\left(L_z - \frac{B}{2}r^2\right). \tag{5.65}$$

Again for a harmonic potential $V(r) = \lambda r^2$, this takes the form

$$H = \frac{\mathbf{p^2}}{2m} + \frac{B'^2}{8m}r^2 - \tilde{\Lambda}L_z \tag{5.66}$$

where, $B' = B\sqrt{1+\frac{8m\lambda}{B^2}(1+\frac{\theta B}{2})}$ and $\tilde{\Lambda} = \frac{B}{2m}+\theta\lambda$. Recognising that the structure of eq.(5.66) is the same as that of eq.(5.19), we can readily write down the ground state wave-function as

$$\begin{aligned} \psi_0(z,\bar{z};\theta) &= \exp\left(-\frac{B'(\theta)}{4}\bar{z}z\right) \\ &= \exp\left(-\frac{1}{4}\bar{z}z\sqrt{B^2+8m\lambda\left(1+\frac{\theta B}{2}\right)}\right) \; ; \\ & \quad |\theta| << 1 \; . \end{aligned} \tag{5.67}$$

This expression clearly reveals the fact that the commutative wave-function has a non-trivial flow in θ of its own, only in the presence of interaction ($\lambda \neq 0$) and the values of both noncommutative wave-functions $\hat{\psi}$, $\hat{\psi}_{sw}$ and the commutative one ψ coincide at $\theta = 0$. One can, in principle, determine the exact expression of this wave-function, valid upto all orders in θ, but we shall not require this here. In fact the wave-function (5.67) or higher angular momentum states $z^l\psi_0(z,\bar{z};\theta)$ can be alternatively determined from perturbation theory applied to each angular momentum sector l for small θ and λ. A point that we would like to emphasise is that the SW map does not map the noncommutative field $\hat{\psi}_{sw}(z,\bar{z};\theta)$ at value θ to the corresponding one at the commutative level $\psi(z,\bar{z};\theta)$; the SW map or equivalently the SW equation (5.34) only relates $\hat{\psi}(z,\bar{z};\theta)$ to $\hat{\psi}(z,\bar{z};\theta=0) = \psi(z,\bar{z};\theta=0)$. Furthermore, the fact that the parameter $\hat{m}_{sw}(\theta)$ (eq.(5.62)) does not reproduce the expression to leading order in θ, derived in the previous chapter eq.(4.30) can be seen to follow from the observation that the parameter m was basically fixed by demanding the form invariance of the Schrödinger action which is equivalent to the stability analysis (in absence of interaction) we have carried out in the previous sections. Also observe that in eq.(4.30) the "renormalised" mass parameter m does not get modified by the interaction term in any way, in contrast to both $\hat{m}_{sw}$ (eq.(5.62)) and $\hat{m}$ (eq.(5.44)). On the other hand, the commutative wave-function ψ in eq.(5.67) gets modified in presence of interaction, as we mentioned above, in such a way that it has a non-trivial flow in θ. This is in contrast to the noncommutative wave-functions $\hat{\psi}$ (eq.(5.50)) and $\hat{\psi}_{sw}$ (eq.(5.59)) which have flows in θ even in absence of interactions. Finally, note that we have three versions of the Hamiltonians here with distinct transformations properties : (i) $\hat{H}$ occuring in eq.(5.2) transforms adjointly under $U(1)_\star$ gauge transformation, (ii) H occuring in eq.(5.64) transforms adjointly under ordinary $U(1)$ gauge transformation and (iii) the Bopp-shifted Hamiltonian $\hat{H}_{BS}$ occuring in eq.(5.2) which, however, does not have any of these transformation properties under either type of gauge transformation as it was constructed just by disentangling the $\star$ product but retaining the noncommutative variables. In this context, it will be worthwhile to remember that in order to have the symmetry under $\star$ gauge transformation we must have noncommutative variables composed through $\star$ product and to have the corresponding symmetry under ordinary gauge transformation, we must replace the noncommutative variables by commutative ones by making use of the SW map apart from disentangling the $\star$ product as was done in the previous chapter [15]. Consequently, the issue of maintaining the gauge invariance/covariance is not relevant here, since we are dealing with $\hat{H}_{BS}$ in this chapter.

5.5 Constructing dualities

In the earlier sections, we have seen how physically equivalent families of noncommuting Hamiltonians can be constructed. In this construction θ simply plays the role of a parameter and subsequently, as the physics does not change, physical quantities can be computed with any value of this parameter. A natural question to pose, therefore, is whether there is any advantage in choosing a specific value of θ, i.e., is there any advantage in introducing noncommutativity in the first place. The motivation for asking this question was already outlined in section 5.1, where it was pointed out that in some existing literature [80], the noncommutative quantum Hall system is considered a paradigm for the fractional quantum Hall effect which, however, requires the presence of interactions. If this interpretation is to be taken seriously a natural possibility that presents itself is that interacting commuting systems may in some approximation be equivalent to a particular non-interacting noncommutative system. If this turns out to be true, it would provide a new rational for the introduction of noncommutativity in quantum Hall systems. In this section we explore this possibility within a very simple setting.

We consider the noncommutative harmonic oscillator moving in a constant magnetic field discussed in section 5.3. After undoing the star product through a Bopp-shift we find the Hamiltonian

$$\begin{aligned}\hat{H}_{BS}(\theta) &= \frac{\mathbf{p}^2}{2M_0} + \frac{\mathbf{x}^2}{2}M_0{\Omega_0}^2 - \Omega_0(\theta)L_z \\ &\quad + \hat{\lambda}\left(\frac{\theta^2}{4}\mathbf{p}^2 + \mathbf{x}^2 - \theta L_z\right) \\ &= \hat{H}_0 + \hat{V} \end{aligned} \tag{5.68}$$

where,

$$\begin{aligned}\frac{1}{2M_0} &= \frac{\left(1+\frac{\bar{B}\theta}{4}\right)^2}{2\hat{m}} \\ \frac{1}{2}M_0{\Omega_0}^2 &= \frac{\bar{B}(\theta)^2}{8\hat{m}(\theta)}\ . \end{aligned} \tag{5.69}$$

To represent equivalent systems, the parameters $\bar{B}$, $\hat{m}$ and $\hat{\lambda}$ are parameterized as in eq.(s) (5.31), (5.44) and (5.48), respectively.

Naively one might argue that when the noncommutative coupling constant $\hat{\lambda}$ becomes small, the interaction term can be neglected on the noncommutative level. However, as this happens when θ becomes large ($\hat{\lambda} \sim 1/\theta$), one sees from the Bopp-shifted equivalent of the Hamiltonian that this is not true due to the θ dependency that is generated by the Bopp-shift. One therefore has to use a different criterion to decide when the interaction term $\hat{V}$ is small and can be neglected. One way is to introduce a norm on the space of operators and check that $\hat{V}$ is small in this norm. The trace norm $\mathrm{tr}(\hat{V}^\dagger\hat{V})$ is divergent and cannot be used; a regularization is required. An obvious alternative candidate to use is the following

$$Z(\theta) = \frac{\mathrm{tr}(\hat{V}^\dagger e^{-\beta\hat{H}_0}\hat{V})}{\mathrm{tr}e^{-\beta\hat{H}_0}}\ . \tag{5.70}$$

Here β plays the role of an energy cutt-off. It is clear that $Z(\theta)$ has all the properties of a norm, in particular $Z(\theta) = 0$ if and only if $\hat{V} = 0$. As remarked before, it is impossible to eliminate $\hat{V}$ completely, however, we can minimize $Z(\theta)$ with respect to θ and in doing this find the value of θ for which the noncommutative non-interacting Hamiltonian $\hat{H}_0$ gives the best approximation to the interacting Hamiltonian. Since the low energy spectrum of $\hat{H}_0$ is biased in the norm (5.70), one can expect that the low energy spectrum of $\hat{H}_0$ would give good agreement with the interacting spectrum, while the agreement will become worse as one moves up in the spectrum of $\hat{H}_0$. Before implementing this program, there is one further complication to take care of. Due to the degeneracy of $\hat{H}_0$ in the angular momentum, the norm (5.70) is still divergent when summing over angular momenta in the trace. However, since $\hat{V}$ is a central potential and subsequently different angular momentum sectors decouple, it is quite sufficient to implement the program above in each angular momentum sector seperately. Under minimization this will give rise to an angular momentum dependent value of θ, giving rise to a lifting in the degeneracy in angular momentum, which is what one would expect in the presence of interactions. To proceed we therefore replace eq.(5.70) by

$$\begin{aligned} Z(\theta,\ell) &= \frac{\mathrm{tr}_\ell(\hat{V}^\dagger e^{-\beta\hat{H}_0}\hat{V})}{\mathrm{tr}_\ell e^{-\beta\hat{H}_0}} \\ &= \sum_{n_-=0}^{\infty} |\langle n_-,\ell|V|n_-,\ell\rangle|^2 e^{-\beta\Omega_0(2n_-+1)} \end{aligned} \tag{5.71}$$

where tr_ℓ denotes that the trace is taken over a fixed angular momentum sector, eq.(5.37) was used and $|n_-,\ell\rangle$ denote the eigenstates of $\hat{H}_0$. This expression can be evaluated straightforwardly to yield

$$\begin{aligned} Z(\theta,\ell) &= \hat{\lambda}^2(\theta)\left[\Gamma(\theta)^2\left(1+\frac{2}{\sinh^2(\beta\Omega_0)}\right)\right. \\ &+ 2\ell\coth(\beta\Omega_0)\Gamma(\theta)\left(\Gamma(\theta)-\theta\right) \\ &+ \left.\ell^2\left(\Gamma(\theta)-\theta\right)^2\right] \end{aligned} \tag{5.72}$$

where,

$$\Gamma(\theta) = \frac{M_0\Omega_0\theta^2}{4} + \frac{1}{M_0\Omega_0}\ . \tag{5.73}$$

For $\beta >> 1/B$, one finds the value of θ that minimizes this expression to be

$$\theta(\ell) = \frac{2(1+\ell)}{B(1+2\ell)} \tag{5.74}$$

at which value $Z(\theta,\ell)\sim\frac{1}{B^2}$, which means that the potential at these values of θ can be treated as a correction of order $1/B$. The eigenvalues of $\hat{H}_0$ at these values of θ are easily evaluated to be

$$
\begin{aligned}
E_{n_-}(\ell) &= 2\Omega_0(\ell)(n_- + 1/2) \\
\Omega_0(\ell) &= \frac{B}{4m}\left(1 + \sqrt{1 + \frac{16\lambda m(\ell+1)}{B^2}}\right) .
\end{aligned} \tag{5.75}
$$

From the above considerations it is clear that the approximation is controlled by $1/B$. One therefore expects eq.(5.75) to agree with the exact result eq.(5.49), at least for the lowest eigenvalues, to order $1/B$. This indeed turns out to be the case. Expanding the lowest eigenvalues of eq.(s) (5.75) and (5.49) to leading order in $1/B$, one finds in both cases

$$
E_0(\ell) = \frac{B}{2m} + \frac{2(\ell+1)\lambda}{B} . \tag{5.76}
$$

This result suggests that it is indeed possible to trade the interactions for noncommutativity, at least in the lowest Landau level and for weak Landau level mixing (large B). It would, of course, be exceedingly naive to immediately extrapolate from the above to realistic quantum Hall systems. However, the above result does suggest a new paradigm for noncommutative quantum Hall systems worthwhile to explore. Within this paradigm interactions get traded, at least in the lowest Landau level, for noncommutativity, explaining the fractional filling fractions and emergence of composite fermions from a new perspective.

5.6 Summary

We have demonstrated how physically equivalent families of noncommutative Hamiltonians can be constructed. This program was explicitly implemented to all orders in the noncommutative parameter in the case of a free particle and harmonic oscillator moving in a constant magnetic field in two dimensions. It was found that this spectrum preserving map coincides with the SW map in the case of no interactions, but not in the presence of interactions. A new possible paradigm for noncommutative quantum Hall systems was demonstrated in a simple setting. In this paradigm an interacting commutative system is traded for a weakly interacting noncommutative system, resulting in the same physics for the low energy sector. This provides a new rational for the introduction of noncommutativity in quantum Hall systems.

Appendix: Eigenvalues and eigenstates of the free and harmonic oscillator Hamiltonians

To solve for the eigenvalues and eigenstates of eq.(5.19), creation and annihilation operators are introduced through the equations

$$
\begin{aligned}
b_x &= \sqrt{\frac{M\Omega}{2}}\left(x + \frac{ip_x}{M\Omega}\right) \quad , \quad b_x^\dagger = \sqrt{\frac{M\Omega}{2}}\left(x - \frac{ip_x}{M\Omega}\right) \\
b_y &= \sqrt{\frac{M\Omega}{2}}\left(y + \frac{ip_y}{M\Omega}\right) \quad , \quad b_y^\dagger = \sqrt{\frac{M\Omega}{2}}\left(y - \frac{ip_y}{M\Omega}\right) .
\end{aligned}
\tag{5.77}
$$

In terms of these operators the Hamiltonian (5.19) takes the form:

$$
H = \Omega\left(b_x^\dagger b_x + b_y^\dagger b_y + 1\right) - i\Omega\left(b_x b_y^\dagger - b_x^\dagger b_y\right) . \tag{5.78}
$$

Now making use of the following set of transformations

$$
\begin{aligned}
b_+ &= \frac{1}{\sqrt{2}}\left(b_x - ib_y\right) \ , \quad b_+^\dagger = \frac{1}{\sqrt{2}}\left(b_x^\dagger + ib_y^\dagger\right) \\
b_- &= \frac{1}{\sqrt{2}}\left(b_x + ib_y\right) \ , \quad b_-^\dagger = \frac{1}{\sqrt{2}}\left(b_x^\dagger - ib_y^\dagger\right)
\end{aligned}
\tag{5.79}
$$

the Hamiltonian (5.78) reads

$$
\begin{aligned}
H &= \Omega\left(b_+^\dagger b_+ + b_-^\dagger b_- + 1\right) - \Omega\left(b_+^\dagger b_+ - b_-^\dagger b_-\right) \\
&= \Omega\left(n_+ + n_- + 1\right) - \Omega\left(n_+ - n_-\right) \\
&= 2\Omega\left(n_- + \frac{1}{2}\right) .
\end{aligned}
\tag{5.80}
$$

Note that the energy spectrum depends only on n_- and is independent of n_+. Therefore, it results in an infinite degeneracy in the energy spectrum. The above cancellation of the terms involving n_+ has taken place since the coefficients of n_+ are equal. This is also true in the limit $\theta = 0$. This feature does not persist in presence of interactions (see section 5.3).

Introducing complex coordinates $z = x + iy$ and $\bar{z} = x - iy$, eq.(5.79) takes the form

$$
b_+ = \frac{1}{2}\sqrt{M\Omega}\left[\bar{z} + \frac{2}{M\Omega}\partial_z\right] \ , \ b_+^\dagger = \frac{1}{2}\sqrt{M\Omega}\left[z - \frac{2}{M\Omega}\partial_{\bar{z}}\right]
$$

$$b_- = \frac{1}{2}\sqrt{M\Omega}\left[z + \frac{2}{M\Omega}\partial_{\bar{z}}\right] \ , \ b_-^\dagger = \frac{1}{2}\sqrt{M\Omega}\left[\bar{z} - \frac{2}{M\Omega}\partial_z\right] \ . \tag{5.81}$$

The ground state wave-function is annihilated by b_-, i.e. $b_-\hat{\psi}(z,\bar{z};\theta) = 0$. This immediately leads to the solution

$$\hat{\psi}_0(z,\bar{z};\theta) = N\exp\left[-\frac{M\Omega}{2}\bar{z}z\right] = N\exp\left[-\frac{\bar{B}}{4\left(1+\frac{\bar{B}\theta}{4}\right)}\bar{z}z\right]. \tag{5.82}$$

Since $\bar{B}(\theta = 0) = B$, the above solution goes smoothly to the commutative result

$$\psi(z,\bar{z}) = N\exp\left[-\frac{B}{4}\bar{z}z\right] \ . \tag{5.83}$$

This state is also annihilated by b_+ and therefore corresponds to zero angular momentum state, as the angular momentum operator $L_3 = (xp_y - yp_x)$ takes the following form

$$L_3 = i\left(b_x b_y^\dagger - b_x^\dagger b_y\right) = \left(b_+^\dagger b_+ - b_-^\dagger b_-\right). \tag{5.84}$$

If this xy-plane is thought to be embedded in $3-d$ Euclidean space $\mathcal{R}^3$, then the other rotational generators L_1 and L_2 obtained by cyclic permutation would result in the standard angular momentum $SU(2)$ algebra

$$[L_i, L_j] = i\epsilon_{ijk}L_k. \tag{5.85}$$

One can, however, define the $SU(2)$ algebra using the creation and annihilation operators alone, which in the cartesian basis (5.77), is given by:

$$\begin{aligned}
J_1 &= \frac{1}{2}\left(b_x^\dagger b_x - b_y^\dagger b_y\right) \\
J_2 &= \frac{1}{2}\left(b_x^\dagger b_y + b_y^\dagger b_x\right) \\
J_3 &= \frac{1}{2i}\left(b_x^\dagger b_y - b_y^\dagger b_x\right)
\end{aligned} \tag{5.86}$$

satisfying $[J_i, J_j] = i\epsilon_{ijk}J_k$. As one can easily verify, by computing the PB(s) of the generators with phase-space variables that J_1 generates rotation in (x, p_x) and (y, p_y) planes, J_y in (x, p_y)

and (y, p_x) planes and J_z in (x, y) and (p_x, p_y) planes. Also note that L_3 is not identical to J_3 but differs by a factor of 2: $L_3 = 2J_3$.

The Casimir operator in terms of J_i representation now becomes

$$\vec{J}^2 = \frac{1}{4}\left(b_+^\dagger b_+ + b_-^\dagger b_-\right)\left(b_+^\dagger b_+ + b_-^\dagger b_- + 2\right) \tag{5.87}$$

with eigenvalues $\vec{J}^2 = \frac{1}{4}\left(n_+ + n_-\right)\left(n_+ + n_- + 2\right)$. Defining $n_+ + n_- = 2j$, the Casimir becomes $\vec{J}^2 = j(j+1)$. Also, if the eigenvalues of J_3 is given by l', then the eigenvalues of L_3 will be given by $n_+ - n_- = 2l' = l \;\epsilon\; \mathcal{Z}$. Note that, like l', j also admits half-integral values. Finally, one can write down the eigenvalues (5.80) as

$$E_{n_-} = \Omega\left(2j - 2l' + 1\right) = \Omega\left(2j - l + 1\right) \tag{5.88}$$

which agrees with [43]. Any arbitrary state can now be obtained by repeated application of $b_\pm^\dagger$ on eq.(5.82) as

$$|n_-, l\rangle \sim \left(b_-^\dagger\right)^{n_-}\left(b_+^\dagger\right)^l \hat{\psi}_0(z, \bar{z}; \theta)\ . \tag{5.89}$$

Chapter 6

Noncommutativity and quantum Hall systems

6.1 Introduction

After investigating noncommutative quantum mechanics in the previous chapter, where we tried to provide a new rationale for introducing noncommutativity in quantum Hall systems in the sense that interactions can be traded with noncommutativity within certain approximation, we now try to present a "complementary" point of view on the impact of noncommutativity stemming from the inter-particle interactions in quantum Hall systems. This issue has recently attracted considerable attention from the point of view of noncommutative quantum mechanics and quantum field theory [29], [65], [30], [71], [78], [80], [92] as it is probably the simplest physical realization of a noncommutative spatial geometry.

Some time ago Dunne, Jackiw and Trugenberger [93], [94] already observed this noncommutativity by noting that in the limit $m \to 0$ the y-coordinate is effectively constrained to the momentum canonical conjugate to the x-coordinate. This result can also be obtained [95, 96] by keeping the mass fixed and taking the limit $B \to \infty$. An alternative point of view is to keep the magnetic field and mass finite, but to project the position coordinates onto the lowest (or higher) Landau level. These projected operators indeed satisfy the commutation relation

($\hbar = e = m = c = 1$) [97]

$$[P_0 x P_0, P_0 y P_0] = \frac{1}{iB} P_0 = \frac{1}{iB} \tag{6.1}$$

where P_0 denotes the projector onto the lowest Landau level, which is also just the identity operator on the projected subspace, as reflected in the last step.

This result allows a simple heuristic understanding of quantum Hall fluids. Recall the elementary uncertainty relation (see *e.g.* [98]) for two noncommuting operators A and B

$$\begin{aligned} (\Delta A)^2 (\Delta B)^2 \quad &\geq \quad \frac{1}{4} \langle i[A, B] \rangle^2 \\ (\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2 \quad , &\qquad (\Delta B)^2 = \langle B^2 \rangle - \langle B \rangle^2 \end{aligned} \tag{6.2}$$

where, $\langle \cdot \rangle$ denotes the normalized expectation value in some state. Using this we note that the noncommutativity of the coordinates implies a lower bound to the area a particle in the lowest Landau level occupies. This bound follows easily from eq.(6.2) to be

$$\Delta A = 4 |\Delta(P_0 x P_0)| |\Delta(P_0 y P_0)| \geq \frac{2}{B} \equiv \Delta A_0 \ . \tag{6.3}$$

Therefore the number of states available in a Landau level is given by:

$$M = \frac{A}{\Delta A_0} = \frac{AB}{2} \ . \tag{6.4}$$

The filling fraction is defined in the usual way as

$$\nu = \frac{N}{M} = \frac{2N}{AB} \tag{6.5}$$

where, N is the number of electrons. For fermions it then follows that at maximum filling of p Landau levels the particles must occupy the minimal allowed area, i.e. $N \Delta A_0 = pA$ and $\nu = p$ with p being an integer. As the next available states are in the higher Landau level, separated in energy by the cyclotron frequency, one expects that the quantum fluid will be incompressible at these values of the filling fraction.

The literature mentioned above does not take into account the effect that interactions between electrons might have on the noncommutativity. In [65], an harmonic potential between two interacting particles was considered, but there the noncommutativity of the center of mass coordinates was investigated, which is again a Landau problem effectively. In particular the analysis of [97] has been done in the absence of any interactions between particles. The conjectured equivalence between a noncommutative $U(1)$ Chern-Simons theory [69, 71] and the composite fermion description for the fractional Hall effect, which is an effective non-interacting theory for the interacting quantum Hall system, urges one to have a better understanding of the relationship between noncommutativity and interactions. A similar picture arises in the much simpler setting of noncommutative quantum Hall systems where it seems as if the fractional quantum Hall effect, associated with an interacting quantum Hall system, can effectively be described by a non-interacting noncommutative quantum Hall system [80], again suggesting an interplay between noncommutativity and interactions. Indeed, keeping the picture of the composite fermion in mind, which replaces electrons interacting through a short ranged repulsive interaction by non-interacting composite fermions moving in a reduced magnetic field, one would expect that the interactions must modify the commutation relation (6.1) as the magnetic field is reduced. A similar conclusion was reached from a completely different point of view in the previous chapter [18]. Here we want to investigate this question in more detail using the approach of [97].

To set the scene, let us consider two interacting particles with the same masses and charges moving in a plane with constant magnetic field perpendicular to the plane. In the symmetric gauge the Hamiltonian is given by ($\hbar = e = m = c = 1$)

$$\begin{aligned} H &= \frac{1}{2}\left(\vec{p}_1 - \vec{A}(\vec{x}_1)\right)^2 + \frac{1}{2}\left(\vec{p}_2 - \vec{A}(\vec{x}_2)\right)^2 + V(|\vec{x}_1 - \vec{x}_2|) \\ A_i(\vec{y}) &= -\frac{B}{2}\epsilon_{ij} y^j,\ B \geq 0\ . \end{aligned} \tag{6.6}$$

Introducing relative and center of mass coordinates through

$$\vec{R} = \frac{1}{2}(\vec{x}_1 + \vec{x}_2) \quad , \quad \vec{r} = \vec{x}_1 - \vec{x}_2 \tag{6.7}$$

the Hamiltonian reduces to

$$\begin{aligned} H &= \frac{1}{4}\left(\vec{P} - 2\vec{A}(\vec{R})\right)^2 + \left(\vec{p} - \frac{1}{2}\vec{A}(\vec{r})\right)^2 + V(|\vec{r}|) \\ \vec{P} &= (\vec{p}_1 + \vec{p}_2), \quad \vec{p} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2)\ . \end{aligned} \tag{6.8}$$

We find that the original problem have got splitted into two decoupled problems. The center of mass motion corresponds to that of a particle with mass $M = 2$ and charge $q = 2e = 2$ moving in a magnetic field B, while the relative motion is that of a particle with reduced mass $\mu = \frac{m}{2} = \frac{1}{2}$ and charge $q = \frac{e}{2} = \frac{1}{2}$ moving in the same magnetic field B and radial potential $V(|\vec{r}|)$. Clearly the cyclotron frequency for both problems is B. The center of mass motion can clearly be analysed as in [97]; projection onto the lowest Landau level will lead to noncommutative coordinates $[P_0XP_0, P_0YP_0] = \frac{1}{2iB}$. Our analysis here concerns the relative motion. This might seem problematic as the potential $V(|\vec{r}|)$ lifts the degeneracy of the Landau levels so that one can apparently no longer think of projection onto Landau levels, and particularly the lowest Landau level. Closer inspection of the argument in [97] reveals, however, that the degeneracy is not essential. Indeed, the only requirement is that the subspace on which is to be projected is infinite dimensional as the noncommutative coordinates can only be realized in this case. A natural generalization of the analysis in [97] would therefore be to identify a low energy infinite dimensional subspace on which to perform the projection. In the case of short range interactions $V(|\vec{r}|)$, for which the interaction energy scale is much less than the cyclotron frequency, which is the situation normally assumed, this can still be done. The reason is that the spectrum for the relative motion will clearly be close to that of the Landau problem for large values of the relative angular momentum as the particles are then well separated. For small values of the angular momentum the potential will have its main effect. However, if the interaction energy scale is much less than the cyclotron frequency, one will still have well separated bands of eigenstates, with the cyclotron frequency being the energy scale determining the separation between bands and the interaction energy scale determining the separation within bands. We can therefore identify an infinite dimensional low energy subspace as the lowest Landau level perturbed by the interaction and proceed to study the commutation relations of the relative coordinates projected onto this subspace.

This chapter is organized in the following way. The general procedure of projection onto the low energy subspace is described in section 6.2. We then apply this procedure to a number of exactly soluble interacting models to obtain insight into the underlying physics in section 6.3. Finally, we conclude with a summary in section 6.4.

6.2 General projection on the low energy sector

We start by recalling a few basic facts about the Landau problem discussed in the previous chapter. A particle moving on a plane, subjected to a perpendicular constant magnetic field B, has a discrete set of energy eigenstates, known as Landau levels, and are labelled as $|n, \ell\rangle$, where n and ℓ are integers labelling the various Landau levels (n) and the degenerate angular momentum eigenstates with integer eigenvalues $\ell(\geq -n)$ within the same Landau level n. We focus on the relative motion of the two particles described by the second part of the Hamiltonian (6.8)

$$H = \left(\vec{p} - \frac{1}{2}\vec{A}(\vec{r})\right)^2 + \tilde{V}(|\vec{r}|). \tag{6.9}$$

From the rotational symmetry this problem can be solved as usual through the separation of variables and the wave functions have the generic form

$$\psi_{n,\ell}(\vec{r}) = \langle\vec{r}|n, \ell\rangle = R_{n,\ell}(r)e^{i\ell\phi} \tag{6.10}$$

where, $R_{n,\ell}$ solves the radial equation

$$\left[-\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\ell^2}{r^2} - \omega_c\ell + \frac{1}{4}\omega_c^2 r^2 + V(|\vec{r}|)\right] R_{n,\ell} = E_{n,\ell}R_{n,\ell} \tag{6.11}$$

n is the principle quantum number and $\omega_c = B/2$ is half of the cyclotron frequency. Under the conditions discussed in section 6.1 the separated bands of eigenstates will be labelled by the principle quantum number, n, while the states within a band will be labelled by the angular mometum ℓ. In particular we assume that the lowest energy states are described by $n = n_0$ (say) and $\ell = 0, 1, 2\ldots$ where we noted from eq.(6.11) that a change in sign of the angular momentum will require an energy of the order of the cyclotron frequency, so that negative angular momenta will not occur in the low energy sector, *i.e.*, we are restricting to the lowest Landau level, perturbed by interactions. We can now construct the projection operator on the low energy sector as

$$P_0 = \sum_{\ell=0}^{\infty} |n_0, \ell\rangle\langle n_0\ell|. \tag{6.12}$$

We now compute the projected relative coordinates

$$P_0 x P_0 = \sum_{l,l'=0}^{\infty} \langle n_0, \ell'|x|n_0, \ell\rangle|n_0, \ell'\rangle\langle n_0, \ell|$$

$$P_0 y P_0 = \sum_{l,l'=0}^{\infty} \langle n_0, \ell' | y | n_0, \ell \rangle | n_0, \ell' \rangle \langle n_0, \ell | \tag{6.13}$$

with

$$\begin{aligned}
\langle n_0, \ell' | x | n_0, \ell \rangle &= \Omega_{\ell',\ell} \left(\delta_{\ell',\ell+1} + \delta_{\ell',\ell-1} \right) \\
\langle n_0, \ell' | y | n_0, \ell \rangle &= -i\Omega_{\ell',\ell} \left(\delta_{\ell',\ell+1} - \delta_{\ell',\ell-1} \right) \\
\Omega_{\ell',\ell} &= \pi \int_0^\infty drr^2 R^*_{n_0,\ell'} R_{n_0,\ell} \, .
\end{aligned} \tag{6.14}$$

The commutator of the relative coordinates then yields

$$[P_0 x P_0, P_0 y P_0] = 2i \sum_{\ell=0}^{\ell=\infty} |\Omega_{\ell,\ell+1}|^2 \left[|n_0, \ell+1\rangle\langle n_0, \ell+1| - |n_0, \ell\rangle\langle n_0, \ell| \right] . \tag{6.15}$$

We now simply have to compute the matrix elements $\Omega_{\ell',\ell}$ to determine the commutator. For some potentials this can be done analytically and exactly, but in most cases one has to resort to approximations. In this regard we note that since the potential has radial symmetry, it will not mix different angular momentum sectors. Within a particular angular momentum sector there is of course no degeneracy of the Landau states, so that one can safely apply perturbation theory to compute the radial wave-functions $R_{n_0,\ell}$ and therefore matrix elements $\Omega_{\ell',\ell}$. Indeed, this corresponds to a $1/B$ expansion. When the interaction is switched off ($V(r) = 0$) the radial wave-functions are those of the Landau problem and this result is easily seen to reduce to eq.(6.1), except for a factor of two. The reason for this is simply that since we are working with the relative coordinates between two particles this commutator should yield in the non-interacting case the minimal area occupied by two particles, which is consistent with eq.(6.1). In contrast to the non-interacting case eq.(6.1), this commutator is in general no longer proportional to P_0. However, since we are dealing with a central potential, the different angular momentum (ℓ) sectors decouple and one can interpret this result as a noncommutative theory with an effective ℓ dependent noncommutative parameter in the same spirit as was done in the previous chapter [18]. As the area occupied by the two particles will increase with increasing relative angular momentum, one can deduce from eq.(s) (6.15) and (6.2) an absolute lower bound to the average area that a particle in the low energy sector may occupy

$$2\Delta A = 4|\Delta(P_0 x P_0)||\Delta(P_0 y P_0)| \geq 4|\Omega_{0,1}|^2. \tag{6.16}$$

The factor of two on the left is required as the right hand side is the average area occupied by two particles, as pointed out earlier.

6.3 Noncommutativity in some soluble models

In this section, we study the noncommutative structure that arises in a number of soluble interacting models to gain deeper insight into the underlying physics.

6.3.1 Harmonic oscillator

We take $V(|\vec{r}|) = \frac{\lambda^2}{4}r^2$. This is not a short range potential and the spectrum will not approach that of the Landau problem for large values of ℓ. Indeed, here one gets a spectrum linearly growing in ℓ (see Fig.6.1) so that one cannot claim that projection onto the lowest principle quantum number will correspond to the lowest energy sector. However, as was pointed out in the introduction, one can in principle project onto any infinite dimensional subspace, not necessarily just the lowest energy, and that is the spirit in which the current calculation is done. The radial equation for the lowest principle quantum number is easy to solve in this case and one obtains:

$$R_{0,\ell} = N_\ell r^\ell \exp(-\frac{1}{4}\sqrt{\omega_c^2+\lambda^2}r^2) \quad ; \quad \ell \geq 0 \tag{6.17}$$

$$N_\ell = \frac{(\omega_c^2+\lambda^2)^{(\ell+1)/2}}{\sqrt{\pi 2^{\ell+1}\Gamma(\ell+1)}}. \tag{6.18}$$

The spectrum is given by:

$$E_{0,\ell} = \ell(\sqrt{\omega_c^2+\lambda^2}-\omega_c) + \sqrt{\omega_c^2+\lambda^2} \tag{6.19}$$

and is linearly growing with increasing ℓ. The spectrum and eigenfunctions for higher quantum numbers can of course also be solved easily and projection onto those subspaces can also be done. The full spectrum is given by $E_{n,\ell} = \sqrt{\omega_c^2+\lambda^2}(2n+1)+\ell(\sqrt{\omega_c^2+\lambda^2}-\omega_c)$, $n = 0, 1, 2\ldots$, $\ell \geq -n$ and is shown in Fig.6.1 for $\omega_c = 1$ and $\lambda = 0.5$. As no new features appear we restrict ourselves here to the solutions with the lowest principle quantum number.

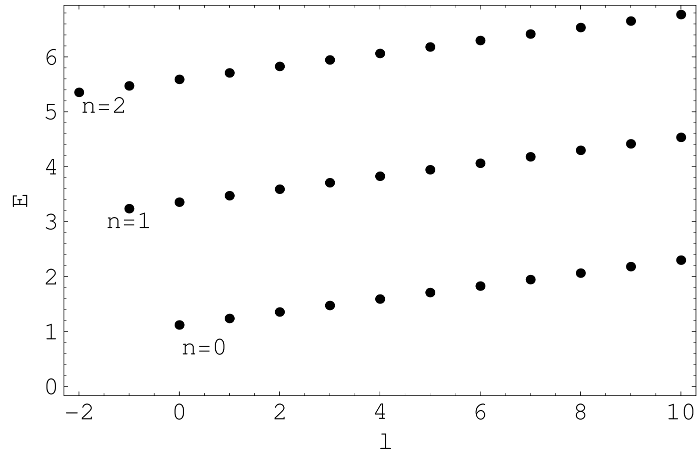

Figure 6.1: The spectrum for the harmonic oscillator potential $\omega_c = 1$ and $\lambda = 0.5$.

The commutator of the relative coordinates can now be evaluated from eq.(6.15) and yields

$$\begin{aligned}
[P_0 x P_0, P_0 y P_0] &= \frac{i}{\sqrt{\omega_c^2 + \lambda^2}} \sum_{\ell=0}^{\ell=\infty} (\ell+1) \left[|n_0, \ell+1\rangle\langle n_0, \ell+1| - |n_0, \ell\rangle\langle n_0, \ell| \right] \\
&= \frac{1}{i\sqrt{\omega_c^2 + \lambda^2}} P_0 \\
&= \frac{2}{iB}\left(1 + \frac{4\lambda^2}{B^2}\right)^{-1/2} . \qquad (6.20)
\end{aligned}$$

In the last line we have noticed that P_0 is just the identity on the projected subspace. As was discussed in general, we note that when the interaction is switched off ($\lambda = 0$), this result differs by a factor of two from eq.(6.1). The first important point to note from this computation is that, generically, the noncommutative parameter is renormalized by the interactions.

We can follow the same heuristic line of reasoning as for the free case to compute the filling fractions at which the interacting quantum Hall fluid behaves incompressibly. The filling factor is $\nu = \frac{N}{M} = \frac{2N}{AB}$. Arguing as in section 6.2, it follows from eq.(6.20) that the average area occupied by a particle is strictly bounded from below by $2\Delta A = 4|\Delta(P_0 x P_0)||\Delta(P_0 y P_0)| \geq 4/B\sqrt{1 + \frac{4\lambda^2}{B^2}} \equiv 2\Delta A_0$. At maximum filling of the p lowest Landau levels (bands) one expects the particles to occupy the minimum allowed area, *i.e.*, $N\Delta A_0 = pA$ and $\nu = p\sqrt{1 + \frac{4\lambda^2}{B^2}}$, p integer. As the next available states are in the next Landau level, which are still separated on an energy scale of the cyclotron frequency under the assumption that the interaction energy scale is much less than the cyclotron frequency, one expects that the quantum fluid will be incompressible at these values of the filling. Note that these filling fractions are larger than the non-interacting values. This is easily understood from the attractive nature of the interaction which, effectively, enhances the magnetic field.

6.3.2 Inverse square potential

Here we take $V(|\vec{r}|) = \frac{2\lambda^2}{r^2}$. This Hamiltonian is very similar in structure to the Hamiltonian of a charged particle moving in a plane and coupled to the gauge potential $A_i = -\frac{\alpha}{r^2}\epsilon_{ij}x^j$ corresponding to a singular flux tube located at the origin, augmented by a harmonic potential. We investigate this case in detail in the next section as it is of particular importance in quantum Hall systems. Taking a cue from the wave function of this Hamiltonian [99], the lowest energy wave functions ($n = 0, \ell \geq 0$) are obtained by making the following ansatz:

$$\psi_{n=0,\ell}(r,\phi) = N_\ell r^{\Lambda(\ell)} e^{i\ell\phi} \exp\left(-\frac{\omega_c}{4} r^2\right) \tag{6.21}$$

where, $\Lambda(\ell)$ is some unknown quantity which will get fixed by eq.(6.11). The solution for $\Lambda(\ell)$, the exact low energy eigenvalues $E_{n=0,\ell}$ and the normalisation constant $N(\ell)$ are given by:

$$\Lambda(\ell) = \left(\ell^2 + 2\lambda^2\right)^{1/2} \tag{6.22}$$

$$E_{n=0,\ell} = \left[\left(\ell^2 + 2\lambda^2\right)^{1/2} - \ell + 1\right]\omega_c \tag{6.23}$$

$$N_\ell = \left[\frac{\omega_c^{\Lambda(\ell)+1}}{\pi 2^{\Lambda(\ell)+1}\Gamma(\Lambda(\ell)+1)}\right]^{1/2} . \tag{6.24}$$

In this case the eigenvalues and eigenfunctions for higher Landau levels can also be solved as in [99]. The full spectrum is given by $E_{n,\ell} = \left[2n + (\ell^2 + 2\lambda^2)^{1/2} - \ell + 1\right]\omega_c,\ n = 0, 1, 2\ldots,$ $\ell \geq 0$ and is shown in Fig.6.2 for $\omega_c = 1$ and $\lambda = 0.5$. The expected features for a short range repulsive interaction can clearly be seen from this graph.

The commutator of the relative coordinates can now be evaluated from eq.(6.15) and yields:

$$[P_0 x P_0, P_0 y P_0] = \frac{2i}{B}\sum_{\ell=0}^{\infty} F(\ell)^2 \left[|0,\ell+1\rangle\langle 0,\ell+1| - |0,\ell\rangle\langle 0,\ell|\right] \tag{6.25}$$

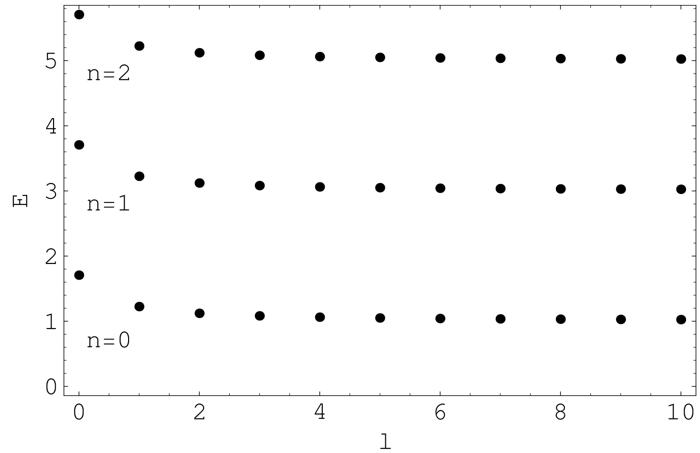

Figure 6.2: The spectrum for the inverse square potential with $\omega_c = 1$ and $\lambda = 0.5$.

where, $F(\ell)$ is given by

$$F(\ell) = \frac{\Gamma\left(\frac{\Lambda(\ell)+\Lambda(\ell+1)+3}{2}\right)}{[\Gamma(\Lambda(\ell)+1)\Gamma(\Lambda(\ell+1)+1)]^{1/2}} \,. \tag{6.26}$$

Note that in this case the right hand side of eq.(6.25) is not proportional to the projection operator P_0 and, as pointed out earlier, one should interpret this as an effective noncommutative theory with an ℓ-dependent renormalized noncommutative parameter.

Note that contrary to what one might naively expect, the lower bound of the area of the particle in angular momentum sector l, given in terms of the quantities $|F(\ell-1)^2 - F(\ell)^2|$, are not monotonically increasing functions of ℓ. To understand this, one must note that this lower bound is only achieved for minimum uncertainty states. The actual area is to be computed from $\langle r^2 \rangle$ in the appropriate eigenstate, which is indeed a monotonically increasing function of ℓ. One therefore concludes that the corresponding expression, evaluated at $\ell = 0$ gives an absolute lower bound. Arguing as before, it follows from eq.(6.25) that the average area occupied by a particle is strictly bounded from below by $2\Delta A = 4|\Delta(P_0 x P_0)||\Delta(P_0 y P_0)| \geq \frac{4F(0)^2}{B} \equiv 2\Delta A_0$. As before the filling fractions at which the fluid is incompressible are $\nu = \frac{p}{F(0)^2}$, p integer. As $F(0)^2 \geq 1$ this yields a fractional filling factor.

6.3.3 Singular magnetic fields

In this section we consider the relative motion of the two particles without any interaction, but with a singular flux tube located at the position of the particles. To obtain the appropriate Hamiltonian [99], we perform a singular gauge transformation in the relative coordinate on the Hamiltonian (6.8). To be precise we perform the gauge transformation $e^{i\alpha\phi} H e^{-i\alpha\phi}$ with $\phi = \tan^{-1}\left(\frac{y}{x}\right)$, with y and x the components of the relative coordinates. Dropping the center of mass part of (6.8), which is not affected by the gauge transformation, the gauge transformed Hamiltonian for the relative coordinate, which corresponds to a singular flux tube inserted at the position of the particles, reads [99]

$$H = \left(\vec{p} - \frac{1}{2}\vec{A}\right)^2 \tag{6.27}$$

where,

$$A_i = -\left(\frac{B}{2} + \frac{2\alpha}{r^2}\right)\epsilon_{ij}x^j,\ B \geq 0\ . \tag{6.28}$$

Written out explicitly the Hamiltonian reads:

$$H = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(i\frac{\partial}{\partial \phi} + \alpha\right)^2 + i\omega_c\frac{\partial}{\partial \phi} + \frac{1}{4}\omega_c^2 r^2 + \alpha\omega_c\ . \tag{6.29}$$

As in the composite fermion paradigm, we choose $\alpha \leq 0$ so that it leads to an effective reduction of the magnetic flux seen by the particles. The low energy eigenfunctions and spectrum are easily found to be [99]:

$$\begin{aligned}
\psi_{0,\ell} &= N_\ell r^{|\ell-\alpha|} e^{i\ell\phi} e^{-\frac{\omega_c r^2}{4}},\ \ell \geq 0 \\
N_\ell &= \left[\frac{\omega_c^{|\ell-\alpha|+1}}{\pi 2^{|\ell-\alpha|+1}\Gamma\left(|\ell-\alpha|+1\right)}\right]^{1/2} \\
E_{0,\ell} &= \omega_c\ .
\end{aligned} \tag{6.30}$$

The commutator of the relative coordinates yields from eq.(6.15)

$$\begin{aligned}
[P_0 x P_0, P_0 y P_0] &= \frac{2i}{B}\sum_{\ell=0}^{\infty}\left(l + |\alpha| + 1\right)\left[|0,\ell+1\rangle\langle 0,\ell+1| - |0,\ell\rangle\langle 0,\ell|\right] \\
&= \frac{2}{iB}\sum_{\ell=0}^{\infty}\left(|\alpha|\delta_{0,\ell} + 1\right)|0,\ell\rangle\langle 0,\ell|.
\end{aligned} \tag{6.31}$$

As before, it follows from eq.(6.31) that the average area occupied by a particle is strictly bounded from below by $2\Delta A = 4|\Delta(P_0xP_0)||\Delta(P_0yP_0)| \geq \frac{4(1+|\alpha|)}{B} \equiv 2\Delta A_0$ and the filling fractions at which the fluid is incompressible are $\nu = \frac{p}{1+|\alpha|}$, p integer. Keeping in mind the phase factor associated with the singular gauge transformation, unchanged statistics requires, as usual, that one must choose $\alpha = 2k$ with k a negative integer. This choice indeed yields the fractional fillings as obtained from the composite fermion picture [100] when appropriate choices of p and k are made.

6.4 Summary

In this chapter, we have discussed the role that interactions play in the noncommutative structure that arises when the relative coordinates of two interacting particles are projected onto the lowest Landau level. The fact that the interactions in general renormalize the noncommutative parameter away from the non-interacting value $\frac{1}{B}$ is transparent from our analysis. The effective noncommutative parameter also depends on the angular momentum in general, as was also found from other considerations in the previous chapter[18]. An heuristic argument, based on the noncommutative coordinates, was given to find the filling fractions at incompressibility and the results are consistent with known results in the case of singular magnetic fields. It should be kept in mind however that this argument was very simplistic as all possible many-body correlations were ignored. Probably due to this oversimplification, this argument cannot explain, for a general short range repulsive interaction, the quantized values of the filling fraction at incompressibility observed in the fractional quantum Hall effect. Indeed, from naive perturbative considerations in the above setting one would expect that the (screened) Coulomb interaction will have only a perturbative effect on the noncommutative parameter and filling fraction, which is certainly not the case. As in other treatments, it is only when one already assumes the existence of composite fermions, as was done in section 6.3.3, that the quantized filling fraction can be explained. The apparently non-perturbative microscopic origin of composite fermions as effective non-interacting degrees of freedom to describe the Coulomb interacting quantum Hall fluid is indeed still an illusive and controversial issue [101].

Chapter 7

Twisted Non-Relativistic Quantum Field Theory

In this chapter, let us take up the issue of Galilean symmetry again in the light of an observation made in the literature recently that the Lorentz symmetry in noncommutative quantum field theory can be restored under a twisted implementation of the Lorentz group[1] [102, 103, 104, 105, 106]. The twist approach was proposed as a way to circumvent the breaking of Lorentz invariance that follows from the choice of a particular noncommutative matrix θ. It has been shown that by invoking the concept of twisted Poincaré symmetry of the algebra of functions on a Minkowski spacetime, the noncommutative spacetime with the commutation relations (2.4), with $\theta_{\mu\nu}$ being a *constant* real antisymmetric matrix, can be interpreted in a Lorentz-invariant way. This is interesting because unlike the earlier scheme studied in chapter 4, this does not make use of SW map, apart from undoing the star product and then terminating the series upto certain order in θ. This is not a very satisfactory standpoint, as, apart from the approximation involved, SW map is not spectrum preserving in an interacting theory as we have seen in chapter 5. For simplicity however, we shall discuss a free theory only in this chapter and see that it can have certain non-trivial consequences in the form of violation of Pauli principle.

[1]So far we have investigated the untwisted formulation of noncommutative quantum field theory in the earlier chapters.

To discuss the basic set up, note that the Poincaré group $\mathcal{P}$ or the diffeomorphism group $\mathcal{D}$ which acts on the noncommutative spacetime $\mathcal{R}^{d+1}$ defines a natural action on smooth functions $\alpha \in C^\infty(\mathcal{R}^{d+1})$ as

$$(g\alpha)(x) = \alpha(g^{-1}x) \tag{7.1}$$

for $g \in \mathcal{P}$ or $\in \mathcal{D}$. However, in general

$$(g\alpha) *_\theta (g\beta) \neq g(\alpha *_\theta \beta) \tag{7.2}$$

showing that the action of the group $\mathcal{P}$ or $\mathcal{D}$ is not an automorphism of the algebra $\mathcal{A}_\theta(\mathcal{R}^{d+1})$, unless one considers the translational sub-group. This violation of Poincaré symmetry in particular is accompanied by the violation of microcausality, spin statistics and CPT theorem in general [23, 107]. These results, which follow from the basic axioms in the canonical (commutative) quantum field theory, are no longer satisfied in presence of noncommutativity in the manner discussed above. Besides, noncommutative field theories are afflicted with infra-red/ultra-violet (IR/UV) mixing. It is however possible for some of these results to still go through even after postulating weaker versions of the axioms used in standard quantum field theory. For example one can consider the proof of CPT theorem given by Alvarez-Gaume et.al [108] where they consider the breaking of Lorentz symmetry down to the subgroup $O(1,1) \times SO(2)$, and replace the usual causal structure, given by the light cone, by the light-wedge associated with the $O(1,1)$ factor of kinematical symmetry group. One can also consider the derivation of CPT and Spin-Statistics theorems by Franco et.al [109] where they invoke only "asymptotic commutativity" i.e. assuming that the fields to be commuting at sufficiently large spatial separations.

All the above problems basically stemmed from the non-invariance (7.2), and therefore it is desirable to look for some way to restore the invariance. Indeed, the invariance can be restored by introducing a deformed coproduct, thereby modifying the corresponding Hopf algebra [102, 110, 111] (see also the prior work of [112]). Since then, this deformed or twisted coproduct has been used extensively in the framework of relativistic quantum field theory, as this approach seems to be quite promising.

The twisted implementation of the Poincaré group leads to two interesting consequences. The first is that there is apparently no longer any IR/UV mixing [113], which indicates that the high and low energy sectors decouple, in contrast to the untwisted formulation. The second striking consequence is an apparent violation of Pauli's principle [114]. This seems to be unavoidable if one wants to restore Poincaré invariance through the twisted coproduct. If there is no IR/UV mixing, one would expect that any violation of Pauli's principle would impact in either the high or low energy sector. Experimental observation at present energies seems to rule out any effect at low energies, therefore if this picture is a true description of nature, we expect that any violation of the Pauli principle can only appear at high energies. It does, therefore, seem worthwhile as a consistency check to investigate this question in more detail and to establish precisely what the possible impact it may have at low energies and why it may not be observable.

One of the quantities where spin statistics manifests itself very explicitly is the two particle correlation function. A way of addressing this issue would therefore be to study the low temperature limit of the two particle correlation function in a twisted implementation of the Poincaré group. Since we are at low energies it would, however, be sufficient to study the NR limit, i.e. the Galilean symmetry. The other motivation for studying the Galilean symmetry is that the second and first quantized formulation in the NR set up is completely equivalent enabling us to extract the probabilistic interpretation quite easily. This is necessary to relate the above mentioned two particle correlation function to joint probability. We therefore need to consider the question whether the Galilean symmetry can also be restored by a suitable twist of the coproduct. This is a non-trivial point and should be looked at carefully, as the Galilean algebra admits a central extension, in the form of mass, unlike the Poincaré case and the boost generator does not have the form of a vector field in spacetime. It may be recalled, in this context, that the presence of spacetime noncommutativity spoils the noncommutative structure under Galileo boost. This question is all the more important because of the observation made by [115] that the presence of spacetime noncommutativity does not spoil the unitarity of the noncommutative theory. However, we have shown that the presence of spacetime noncommutativity in the relativistic case does not have a well defined NR ($c \to \infty$) limit. Furthermore, spacetime noncommutativity gives rise to certain operator ordering ambiguities rendering the extraction of a NR field in the $c \to \infty$ limit non-trivial.

This chapter is organised as follows. The mathematical preliminaries are discussed in section 7.1 where we introduce the concept of Hopf algebra and the deformed or twisted coproduct. Sub-section (7.2.1) of section 7.2 deals with a brief review of the twisted Lorentz transformation properties of quantum spacetime, as was discussed by [102, 116]. This is then extended to the NR case in sub-section (7.2.2). In section 7.3, we then discuss briefly the NR reduction of the Klein-Gordon field to the Schrödinger field in (2 + 1) dimensions in commutative space, which is then used to obtain the action of the twisted Galilean transformation on the Fourier coefficients in section 7.4. We eventually obtain the action of twisted Galilean transformation on NR Schrödinger fields in section 7.5. In section 7.6, we discuss the implications of the subsequent deformed commutation relations on the two particle correlation function of a free gas in two spatial dimensions. We conclude in section 7.7. Finally, we have added an Appendix where we have included some important aspects of Wigner-Inönu group contraction in this context (i.e. Poincaré → Galileo), which we have made use of in the main text.

7.1 Mathematical preliminaries

In this section we give a brief review of the essential results in [114] for the purpose of application in later sections.

Consider a group G that acts on a complex vector space V by a representation ρ. This action is denoted by

$$v \to \rho(g)v \tag{7.3}$$

for $g \in G$ and $v \in V$. Then the group algebra G^* also acts on V. A typical element of G^* is

$$\int dg \, \alpha(g) \, g \quad , \quad \alpha(g) \in \mathcal{C} \tag{7.4}$$

where dg is an invariant measure on G. Its action is

$$v \to \int dg \, \alpha(g) \, \rho(g) \, v \ . \tag{7.5}$$

Both G and G^* act on $V \otimes V$, the tensor product of V's, as well. These actions are usually taken to be

$$v_1 \otimes v_2 \to [\rho(g) \otimes \rho(g)] (v_1 \otimes v_2) = \rho(g)v_1 \otimes \rho(g)v_2 \tag{7.6}$$

and

$$v_1 \otimes v_2 \to \int dg \, \alpha(g) \, \rho(g)v_1 \otimes \rho(g)v_2 \tag{7.7}$$

respectively, for $v_1, v_2 \in V$. In Hopf algebra theory [117, 118], the action of G and G^* on tensor products is defined by the coproduct Δ_0, a homomorphism from G^* to $G^* \otimes G^*$, which on restriction to G gives a homomorphism from G to $G^* \otimes G^*$. This restriction specifies Δ_0 on all of G^* by linearity. Hence, if

$$\begin{aligned} \Delta_0 :\ g &\to \Delta_0(g) \\ \Delta_0(g_1)\Delta_0(g_2) &= \Delta_0(g_1 g_2) \end{aligned} \tag{7.8}$$

we have

$$\Delta_0 \left(\int dg \ \alpha(g) g \right) = \int \ dg \ \alpha(g) \Delta_0(g). \tag{7.9}$$

We now make an elevation. Suppose that V is an algebra $\mathcal{A}$. As $\mathcal{A}$ is an algebra, we have a rule for taking products of elements of $\mathcal{A}$, which means that there exists a multiplication map

$$\begin{aligned} m : \mathcal{A} \otimes \mathcal{A} &\to \mathcal{A} \\ \alpha \otimes \beta &\to m(\alpha \otimes \beta) \end{aligned} \tag{7.10}$$

for $\alpha, \beta \in \mathcal{A}$, the product $\alpha\beta$ being $m(\alpha \otimes \beta)$.

The compatibility of Δ_0 with m is now essential, so that:

$$m \left((\rho \otimes \rho) \Delta_0(g) \, (\alpha \otimes \beta) \right) = \rho(g) m(\alpha \otimes \beta) \ . \tag{7.11}$$

In the Moyal plane, the multiplication denoted by the map m_θ is noncommutative and depends on $\theta^{\mu\nu}$. It is defined by[2]

$$\begin{aligned} m_\theta(\alpha \otimes \beta) &= m_0 \left(e^{-\frac{i}{2}(i\partial_\mu)\theta^{\mu\nu} \otimes (i\partial_\nu)} \alpha \otimes \beta \right) \\ &= m_0 \left(F_\theta \alpha \otimes \beta \right) \end{aligned} \tag{7.12}$$

[2] The signature we are using is $(+, -, -, ...)$.

where, m_0 is the usual point-wise multiplication of two functions. Note that here we have introduced a new twist element F_θ given by

$$\begin{aligned} F_\theta &= e^{-\frac{i}{2}\theta^{\mu\nu}P_\mu\otimes P_\nu} \\ &= e^{-\frac{i}{2}(i\partial_\mu)\theta^{\mu\nu}\otimes(i\partial_\nu)} \quad ; \quad P_\mu = i\partial_\mu. \end{aligned} \tag{7.13}$$

The twist element F_θ changes the coproduct to

$$\Delta_0(g) \to \Delta_\theta(g) = \hat{F}_\theta^{-1}\Delta_0(g)\hat{F}_\theta \tag{7.14}$$

in order to maintain compatibility with m_θ, as can be easily checked. In the case of the Poincaré group, if $\exp(iP{\cdot}a)$ is a translation, we have:

$$\begin{aligned} (\rho\otimes\rho)\Delta_\theta\left(e^{iP\cdot a}\right)e_p\otimes e_q &= (\rho\otimes\rho)\left[\hat{F}_\theta^{-1}(e^{iP\cdot a}\otimes e^{iP\cdot a})\hat{F}_\theta\right] \\ &= e^{i(p+q)\cdot a}e_p\otimes e_q \quad ; \quad (e_p(x) = e^{-ip\cdot x}) \end{aligned} \tag{7.15}$$

while if Λ is a Lorentz transformation

$$(\rho\otimes\rho)\Delta_\theta(\Lambda)e_p\otimes e_q = \left[e^{\frac{i}{2}(\Lambda p)_\mu\theta^{\mu\nu}(\Lambda q)_\nu}e^{-\frac{i}{2}p_\mu\theta^{\mu\nu}q_\nu}\right]e_{\Lambda p}\otimes e_{\Lambda q}\ . \tag{7.16}$$

These relations are derived in [114]. Finally, we mention the action of the coproduct Δ_0 on the elements of a Lie-algebra $\mathcal{A}$. The coproduct is defined on $\mathcal{A}$ by

$$\Delta_0(X) = X\otimes 1 + 1\otimes X. \tag{7.17}$$

Its action on the elements of the corresponding universal covering algebra $\mathcal{U}(\mathcal{P})$ can be calculated through the homomorphism [119] :

$$\Delta_0(XY) = \Delta_0(X)\Delta_0(Y) = XY\otimes 1 + X\otimes Y + Y\otimes X + 1\otimes XY. \tag{7.18}$$

One can also easily check that this action of the coproduct on the Lie-algebra is consistent with the action on the group element defined by

$$\Delta_0(g) = g\otimes g. \tag{7.19}$$

7.2 Transformation properties of tensors under space-time transformation

7.2.1 Lorentz transformation

In this sub-section, we give a brief review of the Lorentz transformation properties in the commutative case to set the scene for the rest of the chapter. This turns out to be essential in understanding the action of the Lorentz generators on any vector or tensor field.

Let us consider an infinitesimal Lorentz transformation

$$x^\mu \to x'^\mu = x^\mu + \omega^{\mu\nu} x_\nu \tag{7.20}$$

where, $\omega^{\mu\nu}$ is an infinitesimal constant ($\omega^{\mu\nu} = -\omega^{\nu\mu}$). Any vector field A_μ under this transformation transforms as

$$A_\mu \to A'_\mu(x') = A_\mu(x) + \omega_\mu{}^\lambda A_\lambda(x)\ . \tag{7.21}$$

The functional change in $A_\mu(x)$ therefore reads

$$\begin{aligned} \delta_0 A_\mu(x) &= A'_\mu(x) - A_\mu(x) \\ &= \omega^{\nu\lambda} x_\nu \partial_\lambda A_\mu(x) + \omega_{\mu\nu} A^\nu \\ &= -\frac{i}{2}\omega^{\nu\lambda} J_{\nu\lambda} A_\mu \end{aligned} \tag{7.22}$$

where, $J_{\nu\lambda} = M_{\nu\lambda} + S_{\nu\lambda}$ are the total Lorentz generators with $M_{\mu\nu}$ and $S_{\mu\nu}$ identified with orbital and spin parts, respectively. This immediately leads to the representation of $M_{\nu\lambda}$

$$M_{\nu\lambda} = i(x_\nu \partial_\lambda - x_\lambda \partial_\nu) = (x_\nu P_\lambda - x_\lambda P_\nu)\,; \quad P_\lambda = i\partial_\lambda\,. \tag{7.23}$$

The representation of $S_{\nu\lambda}$ can be found by making use of the relation $\frac{i}{2}\omega^{\rho\lambda}(S_{\rho\lambda}A)_\mu = \omega_{\mu\nu}A^\nu$ obtained by comparing both sides of eq.(7.22). This leads to

$$(S_{\alpha\beta})_{\mu\nu} = i(\eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\mu\beta}\eta_{\nu\alpha})\,. \tag{7.24}$$

It can now be easily checked that $M_{\mu\nu}$, $S_{\mu\nu}$ and $J_{\mu\nu}$ all satisfy the same homogeneous Lorentz algebra $SO(1,3)$:

$$[M_{\mu\nu}, M_{\lambda\rho}] = i\left(\eta_{\mu\lambda}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\lambda} - \eta_{\nu\lambda}M_{\mu\rho} + \eta_{\nu\rho}M_{\mu\lambda}\right). \tag{7.25}$$

Setting $A_\mu = x_\mu$, where x_μ represents a position coordinate of a spacetime point, yields

$$\delta_0 x_\mu = -\frac{i}{2}w^{\nu\lambda}(M_{\nu\lambda} + S_{\nu\lambda})x_\mu = 0 \tag{7.26}$$

as expected, since the Lie derivative of the "radial" vector field $\vec{X} = x^\mu \partial_\mu$ w.r.t. the "rotation" generators (7.23) $M_{\mu\nu}$ vanishes i.e. $\mathcal{L}_{M_{\mu\nu}}\vec{X} = 0$.

Now we observe that the change in x_μ (not the functional change $\delta_0 x_\mu$ as in eq.(7.22)) defined by

$$\delta x_\mu = x'_\mu - x_\mu = \omega_\mu{}^\nu x_\nu \tag{7.27}$$

can be identified as the action of $S_{\nu\lambda}$ on x_μ

$$\delta x_\mu = \omega_\mu{}^\nu x_\nu = -\frac{i}{2}\omega^{\nu\lambda}\left(S_{\nu\lambda}x\right)_\mu \tag{7.28}$$

with the representation of $S_{\nu\lambda}$ given in eq.(7.24). Using eq.(7.26), one can also obtain the action of $M_{\nu\lambda}$ on x_μ[3]

$$\delta x_\mu = -\frac{i}{2}\omega^{\nu\lambda}M_{\nu\lambda}x_\mu. \tag{7.29}$$

The generalization of this to higher second rank tensors $f_{\rho\sigma}(x) = x_\rho x_\sigma$ is straightforward as

$$\delta\left(x_\lambda x_\sigma\right) = \left(-\frac{i}{2}w^{\mu\nu}M_{\mu\nu}\right)\left(x_\lambda x_\sigma\right) \tag{7.30}$$

since we can write

$$\begin{aligned} M_{\mu\nu}f_{\rho\sigma} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu)f_{\rho\sigma} \\ &= i(f_{\mu\sigma}\eta_{\nu\rho} - f_{\nu\sigma}\eta_{\mu\rho} + f_{\rho\nu}\eta_{\mu\sigma} - f_{\rho\mu}\eta_{\nu\sigma}) \end{aligned} \tag{7.31}$$

[3]Note that $\delta A_\mu = A'_\mu(x') - A_\mu(x) = \omega_\mu{}^\lambda A_\lambda(x)$ is not the functional change and δx_μ in eq.(7.29) is obtained by setting $A_\mu = x_\mu$.

where we have made use of eq.(7.23). This indeed shows the covariant nature of the transformation properties of $f_{\rho\sigma}$.

We now review the corresponding covariance property in the noncommutative case under the twisted coproduct of Lorentz generators [102], [116]. The issue of violation of Lorentz symmetry in noncommutative quantum field theories has been known for a long time, since field theories defined on a noncommutative spacetime obeying the commutation relation (2.4) between the coordinate operators, where $\theta_{\mu\nu}$ is treated as a constant antisymmetric matrix, are obviously not Lorentz invariant. However, a new kind of symmetry known as twisted Poincaré symmetry has been found in [102] under which quantum field theories defined on noncommutative spacetime are still Poincaré invariant.

To generalise to the noncommutative case, first note that the star product between two vectors x_μ and x_ν given as $x_\mu \star x_\nu$ is not symmetric, unlike in the commutative case. One can, however, write this as

$$x_\mu \star x_\nu = x_{\{\mu} \star x_{\nu\}} + \frac{i}{2}\theta_{\mu\nu} \tag{7.32}$$

where the curly brackets $\{\}$ denotes symmetrization in the indices μ and ν. This can be easily generalised to higher ranks, showing that every tensorial object of the form $(x_\mu \star x_\nu \star .....\star x_\sigma)$ can be written as a sum of symmetric tensors of equal or lower rank, so that the basis representation is symmetric. Consequently $f_{\rho\sigma}$ should be replaced by the symmetrized expression $f^\theta_{\rho\sigma} = x_{\{\rho} \star x_{\sigma\}} = \frac{1}{2}(x_\rho \star x_\sigma + x_\sigma \star x_\rho)$, and correspondingly the action of the Lorentz generator should be applied through the twisted coproduct (7.14)

$$\begin{aligned} M^\theta_{\mu\nu} f^\theta_{\rho\sigma} &= M^\theta_{\mu\nu} m_\theta \left(x_\rho \otimes x_\sigma\right) = m_\theta \left(\Delta_\theta \left(M_{\mu\nu}\right) \left(x_\rho \otimes x_\sigma\right)\right) \\ &= i(f^\theta_{\mu\sigma}\eta_{\nu\rho} - f^\theta_{\nu\sigma}\eta_{\mu\rho} + f^\theta_{\rho\nu}\eta_{\mu\sigma} - f^\theta_{\rho\mu}\eta_{\nu\sigma}). \end{aligned} \tag{7.33}$$

In the above equation, $M^\theta_{\mu\nu}$ denotes the usual Lorentz generator, but with the action of a twisted coproduct. In [102], it was shown that $M^\theta_{\mu\nu}(\theta^{\rho\sigma}) = 0$, and

$$M^\theta_{\mu\nu}\left(S_t^2\right) = 0 \; ; \quad (S_t^2 = x_\sigma \star x_\sigma) \tag{7.34}$$

i.e. the antisymmetric tensor $\theta^{\rho\sigma}$ is twisted-Poincaré invariant.

7.2.2 Twisted Galilean Invariance

We extend the results of the earlier sub-section on twisted Poincaré invariance to the corresponding NR case in this sub-section. To demonstrate the need for this, consider the Galilean boost transformation

$$\begin{aligned} t \rightarrow t' &= t \\ x^i \rightarrow x'^i &= x^i - v^i t \end{aligned} \tag{7.35}$$

applied in the noncommutative Galilean spacetime having the following noncommutative structure

$$\left[t, x^i\right] = i\theta^{0i} \quad ; \quad \left[x^i, x^j\right] = i\theta^{ij}. \tag{7.36}$$

In the boosted frame, the corresponding expression is given by

$$\begin{aligned} \left[t', x'^i\right] &= \left[t, x^i\right] = i\theta^{0i} \\ \left[x'^i, x'^j\right] &= i\theta^{ij} + i\left(\theta^{0i}v^j - \theta^{0j}v^i\right). \end{aligned} \tag{7.37}$$

This shows that the noncommutative structure in the primed frame does not preserve its structure unless spacetime noncommutativity disappears i.e. $\theta^{0i} = 0$. Here we show that even in the presence of spacetime noncommutativity the Galilean symmetry can be restored through an appropriate twist. To do this we consider a tangent vector field $\vec{A}(x) = A^\mu(x)\partial_\mu$, in Galilean spacetime. Under Galilean transformations (7.35), we have

$$\begin{aligned} A^i(x) \rightarrow A'^i(x') &= \frac{\partial x'^i}{\partial x^\mu}A^\mu(x) = A^i(x) - v^i A^0(x) \\ A^0(x) \rightarrow A'^0(x') &= A^0(x). \end{aligned} \tag{7.38}$$

From eq.(7.38), it follows that

$$\begin{aligned} \delta_0 A^\mu(x) &= A'^\mu(x) - A^\mu(x) \\ &= iv^j\left(-it\partial_j A^\mu(x) + i\delta^\mu_j A^0(x)\right) \\ &= iv^j K_j A^\mu(x) \end{aligned} \tag{7.39}$$

where,

$$\begin{aligned} K_j A^\mu(x) &= \left(-it\partial_j A^\mu(x) + i\delta_j^\mu A^0(x)\right) \\ &= -tP_j A^\mu(x) + i\delta_j^\mu A^0(x). \end{aligned} \tag{7.40}$$

Setting $A^\mu(x) = x^\mu$[4] we easily see that $K_j x^\mu = 0$, from which we get

$$\delta x^\mu = iv^j t P_j x^\mu = iv^j K_j^{(0)} x^\mu \tag{7.41}$$

where, $K_j^{(0)} = tP_j$. This is the counterpart of eq.(7.29) in the Galilean case. In other words, here $K_j^{(0)}$ plays the same role as $M_{\mu\nu}$ in the relativistic case. Indeed, it can be easily checked that at the commutative level it has its own coproduct action

$$K_j^{(0)} m\left(x^\mu \otimes x^\nu\right) = m\left(\Delta_0\left(K_j^{(0)}\right)\left(x^\mu \otimes x^\nu\right)\right). \tag{7.42}$$

Here $K_j^{(0)}$ is clearly the boost generator $K_j^{(M)}$ (see eq.(7.91) in Appendix) with $M = 0$. Note that with $M \neq 0$, $K_j^{(M)}$ does not have the right coproduct action (7.42). This is also quite satisfactory from the point of view that the noncommutativity of spacetime is an intrinsic property and should have no bearing on the mass of the system inhabiting it. We also point out another dissimilarity between the relativistic and NR case. In the relativistic case, the generators $M_{\mu\nu}$ (eq.(7.23)) can be regarded as the vector field whose integral curve generates the Rindler trajectories, i.e. the spacetime trajectories of uniformly accelerated particle. On the other hand, the vector field associated with the parabolic trajectories of uniformly accelerated particle in the NR case is given by K_i^{NR} (eq.(7.88)), which however cannot be identified with the Galileo boost generator $K_j^{(M)}$ (eq.(7.91)) (see Appendix), unlike $M_{\mu\nu}$ in the relativistic case. At the noncommutative level, the action of the Galilean generator should be applied through the twisted coproduct

$$K_j^{\theta(0)} m_\theta\left(x^\mu \otimes x^\nu\right) = m_\theta\left(\Delta_\theta\left(K_j^{(0)}\right)\left(x^\mu \otimes x^\nu\right)\right). \tag{7.43}$$

Using this and noting $K_j^{(0)} = tP_j$, we have

$$\Delta_\theta\left(K_j^{(0)}\right) = \Delta_0\left(K_j^{(0)}\right) \tag{7.44}$$

[4]Here we identify x^0 to be just the time t, rather than ct.

which eventually leads to

$$\begin{aligned} K_j^{\theta(0)} m_\theta \left(x^\mu \otimes x^\nu\right) &= it\left(x^\mu \delta_j^\nu + \delta_j^\mu x^\nu\right) \\ \Rightarrow K_j^{\theta(0)} m_\theta \left(x^\mu \otimes x^\nu - x^\nu \otimes x^\mu\right) &= 0 \\ \Rightarrow K_j^{\theta(0)} \left(\theta^{\mu\nu}\right) &= 0 \end{aligned} \tag{7.45}$$

i.e. the antisymmetric tensor $\theta^{\mu\nu}$ is invariant under twisted Galilean boost. The complete twisted Galilean invariance of $\theta^{\mu\nu}$ is therefore established since the rest of the Galileo generators have the same form as that of the Poincaré generators, discussed in the previous sub-section. To put it more simply, eq.(7.44) clearly shows that the boost generator is taken care of rather easily and the only non-triviality arises in the restoration of rotational symmetry.

7.3 Non-Relativistic reduction in commutative space

In this section, we discuss the NR reduction ($c \to \infty$) of the Klein-Gordon field to the Schrödinger field in 2+1 dimension[5], as this will be used in the subsequent sections to derive the deformed algebra of the Schrödinger field both in the momentum and in the configuration space. The deformed algebra in the momentum space for the Klein-Gordon field has already been derived in [114]. Therefore it is advantageous to consider the NR limit of such a deformed algebra.

We reintroduce the speed of light 'c' in appropriate places from dimensional consideration to take the $c \to \infty$ limit at the end of the calculation, but we still work in the unit $\hbar = 1$. Let us consider the complex Klein-Gordon field satisfying the Klein-Gordon equation

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2 + m^2c^2\right)\phi(x) = 0 \tag{7.46}$$

which follows from the extremum condition of the Klein-Gordon action

$$S = \int dtd^2\mathbf{x}\left[\frac{1}{c^2}\dot{\phi}^\star\dot{\phi} - \phi'^\star\phi' - c^2m^2\phi^\star\phi\right]. \tag{7.47}$$

[5]The procedure of NR reduction holds for any spacetime dimension.

The Schrödinger field is identified from the Klein-Gordon field by isolating the exponential factor involving rest mass energy and eventually taking the limit $c \to \infty$.

Hence, we set

$$\phi(\vec{x}, t) = \frac{e^{-imc^2 t}}{\sqrt{2m}} \psi(\vec{x}, t) \tag{7.48}$$

which yields from eq.(7.46) the equation

$$-\frac{1}{2m} \nabla^2 \psi = i \frac{\partial \psi}{\partial t} - \frac{1}{2mc^2} \frac{\partial^2 \psi}{\partial t^2}. \tag{7.49}$$

This reduces to the Schrödinger equation of a free positive energy particle in the limit $c \to \infty$. In this limit the action (7.47) also yields the corresponding NR action as

$$S_{NR} = \int dt d^2x \, \psi^\star \left(i\partial_0 + \frac{1}{2m} \nabla^2 \right) \psi \ . \tag{7.50}$$

The complex scalar field $\phi(\mathbf{x})$ can be Fourier expanded as

$$\phi(\vec{x}, t) = \int d\mu(k) c \left[a(k) e_k + b^\dagger(k) e_{-k} \right] \tag{7.51}$$

where, $d\mu(k) = \frac{d^2\vec{k}}{2k_0 (2\pi)^2}$ is the Lorentz invariant measure and $e_k = e^{-ik.x} = e^{-i\left(Et - \vec{k}\cdot\vec{x}\right)}$. The commutation relation between a_k and $a_k^\dagger$[6] can be found by using the well known equal time commutation relations between ϕ and Π_ϕ:

$$\left[a(k), a^\dagger(k') \right] = (2\pi)^2 \frac{2k_0}{c} \, \delta^2 \left(\vec{k} - \vec{k'} \right) \tag{7.52}$$

and likewise for $b(k)$. In order to get the Fourier expansion of the field in the NR case, we substitute eq.(7.48) in eq.(7.51), which in the limit $c \to \infty$ yields

$$\psi(\vec{x}, t) = \int \frac{d^2\vec{k}}{(2\pi)^2} \frac{\tilde{c}(k)}{\sqrt{2m}} \tilde{e}_k = \int \frac{d^2\vec{k}}{(2\pi)^2} c(k) \tilde{e}_k \tag{7.53}$$

where, $\tilde{e}_k = e^{-i\frac{|\vec{k}|^2 t}{2m}} e^{i\vec{k}\cdot\vec{x}}$, $\tilde{c}(k) = \lim_{c\to\infty} a(k)$ and $c(k) = \frac{1}{\sqrt{2m}} \tilde{c}(k)$ are the Schrödinger modes. As in eq.(7.49), only the positive energy part survives in the $c \to \infty$ limit, so that this limit effectively projects the positive frequency part. The commutation relation (7.52) reduces in the

[6] Note that $k^\mu = \left(\frac{E}{c}, \vec{k} \right)$.

NR limit ($c \to \infty$) to

$$
\begin{aligned}
\left[\tilde{c}(k), \tilde{c}^\dagger(k')\right] &= (2\pi)^2 2m \; \delta^2\left(\vec{k} - \vec{k'}\right) \\
\left[c(k), c^\dagger(k')\right] &= (2\pi)^2 \; \delta^2\left(\vec{k} - \vec{k'}\right) .
\end{aligned} \tag{7.54}
$$

From eq.(s) (7.53) and (7.54), we obtain

$$
\left[\psi(\vec{x}, t) \; , \; \psi^\dagger(\vec{y}, t)\right] = \delta^2\left(\vec{x} - \vec{y}\right) . \tag{7.55}
$$

7.4 Action of twisted Galilean transformation on Fourier coefficients

Let us consider the Fourier expansion of the relativistic scalar field $\phi(\vec{x}, t)$

$$
\phi(\vec{x}, t) = \int d\mu(k) c \tilde{\phi}(k) e_k \; . \tag{7.56}
$$

Here we have deliberately suppressed the negative frequency part as it does not survive in the NR limit $c \to \infty$, as we have seen in the previous section. Considering the action of the Poincaré group elements on ϕ, we get

$$
\rho(\Lambda_c)\phi = \int d\mu(k) c \, \tilde{\phi}(k) e_{\Lambda_c k} = \int d\mu(k) c \, \tilde{\phi}(\Lambda_c^{-1} k) e_k \tag{7.57}
$$

$$
\rho\left(e^{iP\cdot a}\right)\phi = \int d\mu(k) c \, e^{ik\cdot a} \tilde{\phi}(k) e_k \; . \tag{7.58}
$$

Thus the representation $\tilde{\rho}$ of the Poincaré group on $\tilde{\phi}(k)$ is specified by

$$
\begin{aligned}
\left(\tilde{\rho}(\Lambda_c)\tilde{\phi}\right)(k) &= \tilde{\phi}(\Lambda_c^{-1} k) \\
\left(\tilde{\rho}\left(e^{iP\cdot a}\right)\tilde{\phi}\right)(k) &= e^{ik\cdot a}\tilde{\phi}(k) \; .
\end{aligned} \tag{7.59}
$$

Here homogeneous Lorentz transformations have been labeled by Λ_c. The corresponding Galilean transformations will be labeled by Λ_∞ in the $c \to \infty$ limit.

If χ is another scalar field, with Fourier expansion given by

$$
\chi(\vec{x}, t) = \int d\mu(q) c \, \tilde{\chi}(q) e_q \tag{7.60}
$$

the tensor product of fields ϕ and χ is given by

$$\phi \otimes \chi = \int d\mu(k) d\mu(q) c^2 \, \tilde{\phi}(k) \tilde{\chi}(q) e_k \otimes e_q \ . \tag{7.61}$$

Using eq.(7.16), one obtains the action of the twisted Lorentz transformation on the above tensor product of the fields

$$\Delta_\theta(\Lambda_c)(\phi \otimes \chi) = \int d\mu(k) d\mu(q) c^2 \, \tilde{\phi}(\Lambda_c^{-1}k) \tilde{\chi}(\Lambda_c^{-1}q) e^{\frac{i}{2}k_\mu \theta^{\mu\nu} q_\nu} e^{-\frac{i}{2}(\Lambda_c^{-1}k)_\alpha \theta^{\alpha\beta}(\Lambda_c^{-1}q)_\beta} \left(e_k \otimes e_q\right) . \tag{7.62}$$

Substituting eq.(7.48) in the above equation, one can write the corresponding action of the twisted Lorentz transformations on the tensor product of fields ψ and ξ (here ξ is the counterpart of ψ for the field χ as in eq.(7.48)) as

$$\begin{aligned} \Delta_\theta(\Lambda_c)\left(\psi \otimes \xi\right) &= \int d\mu(k) d\mu(q) 2mc^2 \, \tilde{\phi}(\Lambda_c^{-1}k) \tilde{\chi}(\Lambda_c^{-1}q) e^{\frac{i}{2}k_i \theta^{ij} q_j} e^{-\frac{i}{2}(\Lambda_c^{-1}k)_l \theta^{ln}(\Lambda_c^{-1}q)_n} \\ &\qquad \times e^{-2iO(\frac{1}{c^2},\ldots.)} \left(\tilde{e}_k \otimes \tilde{e}_q\right) . \end{aligned} \tag{7.63}$$

Note that we have set $\theta^{0i} = 0$ in the right hand side of the above equation. The underlying reason is that the substitution (7.48) can be carried out only in the absence of spacetime non-commutativity ($\theta^{0i} = 0$) as this removes any operator ordering ambiguities in eq.(7.48). This should not, however, be regarded as a serious restriction as theories with spacetime noncommutativity do not represent a low energy limit of string theory [107, 120, 121]

Hence in the limit $c \to \infty$, we can deduce the action of the twisted Galilean transformations (Λ_∞) on tensor products of the NR fields:

$$\Delta_\theta(\Lambda_\infty)\left(\psi \otimes \xi\right) = \int \frac{d^2\vec{k} d^2\vec{q}}{(2\pi)^4} \tilde{\psi}(\Lambda_\infty^{-1}k) \tilde{\xi}(\Lambda_\infty^{-1}q) e^{\frac{i}{2}mv_1\theta(k_2 - q_2)} \left(\tilde{e}_k \otimes \tilde{e}_q\right) \ . \tag{7.64}$$

Here we have considered a boost along the x^1 direction with velocity v_1 and $\tilde{\psi}(k) = \lim_{c\to\infty} \tilde{\phi}(k)$, $\tilde{\xi}(q) = \lim_{c\to\infty} \tilde{\chi}(q)$.

From the above, one can deduce the action of the twisted Galilean transformations (Λ_∞) on the Fourier coefficients of the NR fields

$$\Delta_\theta(\Lambda_\infty)\left(\tilde{\psi} \otimes \tilde{\xi}\right)(k, q) = \tilde{\psi}\left(\Lambda_\infty^{-1}k\right) \tilde{\xi}\left(\Lambda_\infty^{-1}q\right) e^{\frac{i}{2}mv_1\theta(k_2 - q_2)} \tag{7.65}$$

One can now easily generalise the above result for the case of any arbitary direction of boost as

$$\Delta_\theta(\Lambda_\infty)\left(\tilde{\psi} \otimes \tilde{\xi}\right)(k, q) = \tilde{\psi}\left(\Lambda_\infty^{-1}k\right) \tilde{\xi}\left(\Lambda_\infty^{-1}q\right) e^{\frac{i}{2}m\theta\vec{v}\times(\vec{k}-\vec{q})} \ . \tag{7.66}$$

7.5 Quantum Fields

In this section, we discuss the action of twisted Galilean transformation on NR Schrödinger fields. A free relativistic complex quantum field ϕ of mass m can be expanded in the noncommutative plane (suppressing the negative frequency part) as

$$\phi(\vec{x},t) = \int d\mu(k) c\, d(k) e_k \ . \tag{7.67}$$

This is just the counterpart of eq.(7.51) where $a(k)$ has been replaced by $d(k)$[7].

The deformation algebra involving $d(k)$ has already been derived in [114]. Here, we derive the deformation algebra for the NR case. The NR limit of the complex Klein-Gordon field has already been discussed in the earlier section and the expansion is the following:

$$\psi(\vec{x},t) = \int \frac{d^2\vec{k}}{(2\pi)^2}\frac{\tilde{u}(k)}{\sqrt{2m}}\tilde{e}_k = \int \frac{d^2\vec{k}}{(2\pi)^2} u(k)\tilde{e}_k \ \ ; \ \ u(k) = \frac{1}{\sqrt{2m}}\tilde{u}(k) \tag{7.68}$$

where, $\tilde{u}(k) = \lim_{c\to\infty} d(k)$.

Note that $\tilde{c}(k), c(k)$ are the limits of the operators $\tilde{u}(k), u(k)$ respectively in the limit $\theta^{\mu\nu} = 0$, and they satisfy the relations (7.54). We now argue that such relations are incompatible for $\theta^{\mu\nu} \neq 0$. Rather, $u(k)$ and $u^\dagger(k)$ fulfill certain deformed relations which reduce to eq.(7.54) for $\theta^{\mu\nu} = 0$.

Suppose that

$$u(k)u(q) = \tilde{T}_\theta(k,q)u(q)u(k) \tag{7.69}$$

where, $\tilde{T}_\theta$ is a $\mathcal{C}$-valued function of k and q yet to be determined. The transformations of $u_k u_l = (u\otimes u)(k,l)$ and $u_l u_k$ are determined by Δ_θ. Applying Δ_θ on eq.(7.69) and using eq.(7.65), we get the following[8]:

$$u\left(\Lambda_\infty^{-1}k\right) u\left(\Lambda_\infty^{-1}q\right) e^{\frac{i}{2}mv\theta(k_2-q_2)} = \tilde{T}_\theta(k,q) u\left(\Lambda_\infty^{-1}q\right) u\left(\Lambda_\infty^{-1}k\right) e^{\frac{i}{2}mv\theta(q_2-k_2)}. \tag{7.70}$$

Using eq.(7.69) again in the left hand side of eq.(7.70), we get:

$$\tilde{T}_\theta\left(\Lambda_\infty^{-1}k, \Lambda_\infty^{-1}q\right) = \tilde{T}_\theta(k,q) e^{-imv\theta(k_2-q_2)}. \tag{7.71}$$

[7]Note that $a(k) = \lim_{\theta\to 0} d(k)$.

[8]Without loss of generality, we consider the boost to be along the x^1 direction for calculational convenience. Also we set $v_1 = v$.

Note that this equation can also be obtained from the corresponding relativistic result [114] in the $c \to \infty$ limit provided one takes $\theta^{0i} = 0$ right from the beginning, otherwise the exponential factor become rapidly oscillating in the $c \to \infty$ limit, yielding no well defined NR limit. Thus in the absence of spacetime noncommutativity one has an appropriate NR limit and the above mentioned operator ordering ambiguities can be avoided.

The solution of eq.(7.71) is[9]

$$\tilde{T}_\theta(k, q) = \eta e^{ik_i\theta^{ij}q_j} \ ; \quad (i, j = 1, 2) \tag{7.72}$$

where η is a Galilean-invariant function and approaches the value ± 1 for bosonic and fermionic fields respectively in the limit $\theta = 0$[10]. Substitution of eq.(7.72) in eq.(7.69) yields

$$u(k)u(q) = \eta e^{ik_i\theta^{ij}q_j} u(q)u(k). \tag{7.73}$$

The adjoint of eq.(7.73) gives:

$$u^\dagger(k)u^\dagger(q) = \eta e^{ik_i\theta^{ij}q_j} u^\dagger(q)u^\dagger(k). \tag{7.74}$$

Finally the creation operator $u^\dagger(q)$ carries momentum $-q$, hence its deformed relation reads:

$$u(k)u^\dagger(q) = \eta e^{-ik_i\theta^{ij}q_j} u^\dagger(q)u(k) + (2\pi)^2\delta^2(k - q). \tag{7.75}$$

The above structure of algebra (7.73, 7.74, 7.75) can be understood more easily by using the twisted projection operator P_θ [11] (first introduced in [122]) [123].

Now using eq.(s) (7.73) and (7.75), one can easily obtain the deformation algebra involving the NR fields $\psi(x)$ in the configuration space:

$$\begin{aligned} \psi(x)\psi(y) &= \int d^2x'd^2y'\Gamma_\theta(x, y, x', y')\psi(y')\psi(x') \quad ; \quad \theta \neq 0 \\ \psi(x)\psi(y) &= \eta\psi(y)\psi(x) \quad ; \quad \theta = 0 \end{aligned} \tag{7.76}$$

[9]Note that the NR form of the twist element also appears in [124].

[10]The value of η can be actually taken to be ± 1 for bosonic and fermionic fields for all $\theta^{\mu\nu}$ [114]. An exactly similar NR reduction of the Dirac equation can also be done for the fermionic case.

[11]$P_\theta = F_\theta^{-1} P_0 F_\theta$, where P_0 is the usual projection operator for a two particle system which projects onto the symmetric (anti-symmetric) sub-space describing bosonic (fermionic) statistics.

$$\psi(x)\psi^\dagger(y) = \int d^2x'd^2y'\Gamma_\theta(x,y,x',y')\psi^\dagger(y')\psi(x') + \delta^2(\vec{x}-\vec{y}) \;\; ; \;\; \theta \neq 0$$
$$\psi(x)\psi^\dagger(y) = \eta\psi^\dagger(y)\psi(x) + \delta^2(\vec{x}-\vec{y}) \quad ; \quad \theta = 0 \qquad (7.77)$$

where,

$$\Gamma_\theta(x,y,x',y') = \frac{\eta}{(2\pi)^2} exp\left(\frac{i}{\theta}\left[(x_1'-x_1)(y_2-y_2') - (x_2'-x_2)(y_1-y_1')\right]\right). \qquad (7.78)$$

Note at this stage that in momentum space, the twisted fermions still satisfy $u(k)u(k) = 0$ as follows from (7.73), unlike what happens in ordinary configuration space as $\psi(x)\psi(x) \neq 0$. This indicates that two identical twisted fermions cannot occupy the same slot in momentum space as happens for ordinary fermions, but can occupy the same position in configuration space for $\theta \neq 0$ and can therefore give rise to violation of Pauli's exclusion principle. We take up this issue in the next section.

7.6 Two particle correlation function

In this section, the computation of the two particle correlation function $\frac{1}{Z}\langle r_1, r_2|e^{-\beta H}|r_1, r_2\rangle$ for a free gas in 2+1 dimensions using the canonical ensemble is performed, where Z is the canonical partition function and H is the NR Hamiltonian. This function tells us what the probability is to find particle two at position r_2, given that particle one is at r_1, i.e. it measures two particle correlations. The relevant two particle state is given by

$$\begin{aligned} |r_1, r_2\rangle &= \hat{\psi}^\dagger(r_1)\hat{\psi}^\dagger(r_2)|0\rangle \\ &= \int \frac{dq_1}{(2\pi)^2}\frac{dq_2}{(2\pi)^2} e^*_{q_1}(r_1)e^*_{q_2}(r_2)u^\dagger(q_1)u^\dagger(q_2)|0\rangle \, . \end{aligned} \qquad (7.79)$$

The two particle correlation function can therefore be written as

$$\langle r_1, r_2|e^{-\beta H}|r_1, r_2\rangle = \int dk_1 dk_2 e^{-\frac{\beta}{2m}(k_1^2+k_2^2)}|\langle r_1, r_2|k_1, k_2\rangle|^2 \qquad (7.80)$$

where we have introduced a complete set of momentum eigenstates $|k_1, k_2\rangle$.

Using eq.(7.75) and noting that

$$|k_1, k_2\rangle = u^\dagger(k_1)u^\dagger(k_2)|0\rangle \qquad (7.81)$$

we finally obtain

$$C(r) \equiv \frac{1}{Z}\langle r_1, r_2|e^{-\beta H}|r_1, r_2\rangle \;\; = \;\; \frac{1}{A^2}\left(1 \pm \frac{1}{1+\frac{\theta^2}{\lambda^4}} e^{-2\pi r^2/(\lambda^2(1+\frac{\theta^2}{\lambda^4}))}\right) \tag{7.82}$$

where, A is the area of the system and λ is the mean thermal wavelength given by

$$\lambda \;\; = \;\; \left(\frac{2\pi\beta}{m}\right)^{1/2} \quad ; \quad \beta = \frac{1}{k_B T} \tag{7.83}$$

and $r = r_1 - r_2$. The plus and the minus signs indicate bosons or fermions.

Although this calculation was done in 2+1 dimensions, it is clear that the result generalizes to higher dimensions by replacing θ^2 by an appropriate sum of $(\theta^{ij})^2$. The conclusions made below, based on the general structure of the correlation function, will therefore also hold in higher dimensions.

Expectedly, this result reduces to the standard (untwisted) result in the limit $\theta \to 0$ [125]. Furthermore it is immediately clear that when $\lambda >> \sqrt{\theta}$, i.e., in the low temperature limit, there is virtually no deviation from the untwisted result as summarized in figure 7.1. This is reassuring as it indicates that the implied violation of Pauli's principle will have no observable effect at current energies. Indeed, keeping in mind that $\sqrt{\theta}$ is probably at the Planck length scale any deviation will only become apparent at very high temperatures, where the NR limit is invalidated. Note, however, that in contrast to the untwisted case the correlation function for fermions does not vanish in the limit $r \to 0$. Thus, there is a finite probability that fermions may come very close to each other[12]. This is most clearly seen from the exchange potential $V(r) = -k_B T \log C(r)$ [125, 126] shown in figure 7.2. This clearly demonstrates the change from a hardcore potential in the untwisted case to a soft core potential in the twisted case. This may have possible implications in astrophysical scenarios, although it is dubious that these densities are even reachable in this case. In any case the assumptions we made here are certainly violated at these extreme conditions and a much more careful analysis is required to investigate the high temperature and high density consequences of the twisted statistics. Another interesting point to note from figure 7.2 is that the twisted statistics has, even at these unrealistic values of $\frac{\theta}{\lambda^2}$,

[12]It should be noted however that this probability is determined by θ and therefore is very small, probably rendering it undetectable.

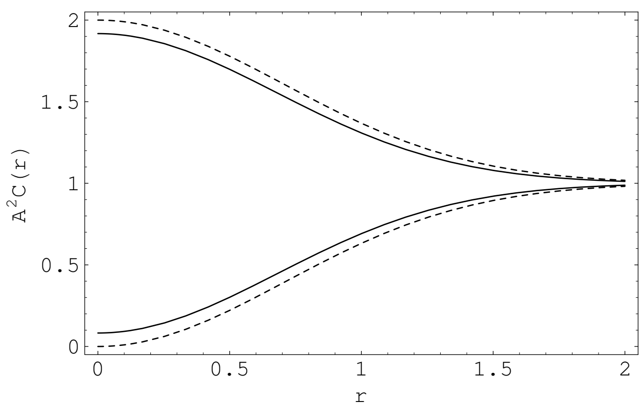

Figure 7.1: Two particle correlation function $C(r)$. The upper two curves is the bosonic case and the lower curves the fermionic case. The solid line shows the twisted result and the dashed line the untwisted case. This is shown for a schematic value of $\frac{\theta}{\lambda^2} = 0.3$. The separation r is measured in units of the thermal length λ.

virtually no effect on the bosonic correlation function at short separation probably suggesting that there will be no observable effect in Bose-Einstein condensation experiments. These results may also have interesting consequences for condensed matter systems such as the quantum Hall effect where the noncommutative parameter is related to the inverse of the magnetic field.

7.7 Summary

We have shown that the noncommutative parameter is twisted Galilean invariant even in presence of spacetime noncommutativity. This is significant in view of the fact that the usual Galilean symmetry is spoiled in presence of spacetime noncommutativity.

We have then derived the deformed algebra of the Schrödinger field in configuration and momentum space. This was done by studying the action of the twisted Galilean symmetry on the Schrödinger field as obtained from a NR reduction of the Klein-Gordon field. The absence of any spacetime noncommutativity had to be considered here as otherwise one cannot define a proper NR limit.

The possible consequences of this deformation in terms of a violation of the Pauli principle

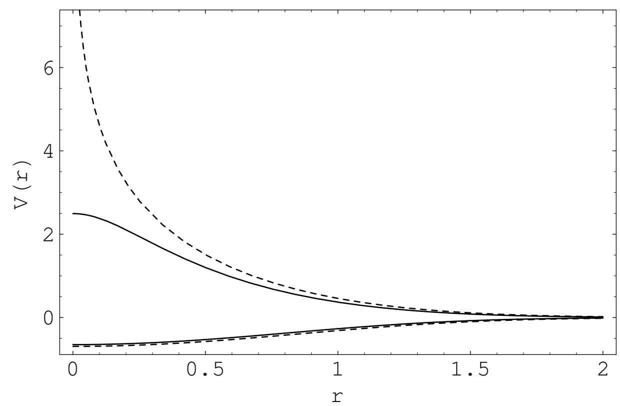

Figure 7.2: Exchange potential $V(r)$ measured in units of k_BT. The irrelevant additive constant has been set zero. The upper two curves is the fermionic case and the lower curves the bosonic case. The solid line shows the twisted result and the dashed line the untwisted case. This is shown for a schematic value of $\frac{\theta}{\lambda^2} = 0.3$. The separation r is measured in units of the thermal length λ.

was studied by computing the two particle correlation function. From this computation, one can infer that any possible effect of the twisted statistics only show up at very high energies, while the effect at low energies should be very small, consistent with current experimental observations. Whether this effect will eventually be detectable through some very sensitive experiment is an open and enormously challenging question.

Appendix: A brief derivation of Wigner-In$\ddot{o}$nu group contraction of Poincaré group to Galilean group

Here we summarise the well known Wigner-In$\ddot{o}$nu group contraction from Poincaré to Galilean algebra in order to highlight some of the subtleties involved, as these have direct bearings on the issues discussed in section (7.2).

To begin with let us consider a particle undergoing uniform acceleration 'a', along the x direction, measured in the instantaneous rest frame of the particle. A typical spacetime Rindler trajectory is given by the hyperbola

$$x^2 - c^2t^2 = \rho^2 \tag{7.84}$$

so that the acceleration $A(t)$ w.r.t the fixed observer with the above associated coordinates (t, x) measured at time t is,

$$A(t) = \frac{dV(t)}{dt} \quad = \quad \frac{c^2}{x}\left(\frac{\rho^2}{x^2}\right).$$

Since the frame (x, t) appearing in eq.(7.84) coincides with that of the fixed observer at time $t = 0$, we must have

$$\Rightarrow a = A(t = 0) \quad = \quad \frac{c^2}{\rho} \tag{7.85}$$

where, ρ is the distance measured at that instant from the origin. To take the NR limit, we have to take both $c \to \infty$ and $\rho \to \infty$ such that $\frac{c^2}{\rho} = a$ is held constant. For example, the corresponding non-relativistic expression $\bar{x}$ for the distance travelled by the particle in time t is obtained by identifying

$$\bar{x} = \lim_{c\to\infty\rho\to\infty}(x - \rho) = \frac{1}{2}at^2 \tag{7.86}$$

which reproduces the standard result.

Now let us consider the Lorentz generator along the x direction $M_{01} = i\left(x_0\partial_1 - x_1\partial_0\right)$. This can be rewritten in terms of $\bar{x}$ using eq.(7.86),

$$\begin{aligned} M_{01} \quad &= \quad ic\left(t\frac{\partial}{\partial\bar{x}} + \frac{1}{a}\left(1 + \frac{\bar{x}}{\rho}\right)\frac{\partial}{\partial t}\right) \\ &= \quad cK_1. \end{aligned} \tag{7.87}$$

Note that K_1 by itself does not have any c dependence, the NR limit of K_1 can thus be obtained by just taking the limit $\rho \to \infty$, which yields

$$K_1^{NR} = \lim_{\rho\to\infty} K_1 = t\frac{\partial}{\partial\bar{x}} + \frac{1}{a}\frac{\partial}{\partial t}. \tag{7.88}$$

Although this vector field indeed generates the integral curve in the t, $\bar{x}$ plane which is a parabola given by eq.(7.86), it can not be identified with the Galileo boost generator because

$$\left[K_i^{NR}, K_j^{NR}\right] \sim \left(P_i - P_j\right). \tag{7.89}$$

The Galilean algebra on the other hand is obtained by taking the limit $c \to \infty$ of the commutators involving boost in the following way:

$$\begin{aligned}
\left[\bar{K_1}, \bar{K_2}\right] &= \lim_{c\to\infty} \frac{1}{c^2}\left[M_{01}, M_{02}\right] = \lim_{c\to\infty} \frac{1}{c^2} M_{12} = 0 \\
\left[P_1, \bar{K_1}\right] &= \lim_{c\to\infty} \frac{1}{c}\left[P_1, M_{01}\right] = \lim_{c\to\infty} \frac{i}{c^2} P_0 = iM \\
\left[\bar{K_1}, J\right] &= \lim_{c\to\infty} \frac{1}{c}\left[M_{01}, M_{12}\right] = i\bar{K_2}
\end{aligned} \tag{7.90}$$

where M is identified as the mass. The rest of the commutators have the same form as that of Poincaré algebra. This is nothing but the famous Wigner-Inönu group contraction, demonstrated here in construction of the Galilean algebra as a suitable limit of the Poincaré algebra. A simple inspection, at this stage, shows the following form of the Galileo boost generators

$$\bar{K}_i = K_i^{(M)} = it\frac{\partial}{\partial \bar{x}_i} + M\bar{x}_i \tag{7.91}$$

Clearly the rest of the generators in Galilean algebra have the same form as Poincaré algebra. For completeness we enlist the full Galilean algebra in (2 +1) dimension:

$$\begin{aligned}
\left[K_i^{(M)}, K_j^{(M)}\right] &= [P_i, P_j] = [P_i, H] = [J, H] = 0 \\
\left[P_i, K_j^{(M)}\right] &= i\delta_{ij} M \\
[P_i, J] &= i\epsilon_{ij} P_j \\
\left[K_i^{(M)}, J\right] &= i\epsilon_{ij} K_j^{(M)} \\
[P_i, M] &= [H, M] = [J, M] = \left[K_i^{(M)}, M\right] = 0.
\end{aligned} \tag{7.92}$$

Finally note that, here the mass M plays the role of central extension of the centrally extended Galilean algebra.

Chapter 8

Conclusions

The main goal of this thesis is to study some aspects of noncommutative quantum mechanics, untwisted and twisted formulations of noncommutative quantum field theory and applications.

There are different settings for noncommutative field theories. The one that has been most used in all recent applications is based on the so-called Moyal (star) product in which for all calculational purposes (differentiation, integration, etc), the spacetime coordinates are treated as ordinary (commutative) variables and noncommutativity enters into play in the way in which fields are multiplied.

We have first given a brief review of the star product formalism in the thesis. Then we have moved on to discuss a general method of obtaining both spacetime and space-space noncommuting structures in various models in particle mechanics exhibiting reparametrization symmetry. A change of variables has been derived using gauge/reparametrization symmetry transformations which relates the commuting algebra in the conventional gauge to a noncommuting algebra in a non-standard gauge.

The role played by the SW map has been investigated in this work. The map has been used to obtain an effective $U(1)$ gauge invariant Schrödinger action upto order θ (starting from a $U(1)_\star$ gauge invariant noncommutative Schrödinger action) followed by wave-function and mass renormalization. The effect of noncommutativity on the mass parameter appears naturally in our analysis. Another interesting point that we observe is that the external magnetic field has to be static and uniform in order to get a canonical form of Schrödinger equation upto θ-corrected terms, so that a natural probabilistic interpretation emerges. The Galilean symmetry of the model is next investigated where the translation and the rotation generators are seen to form a closed Euclidean sub-algebra of Galilean algebra. However, the boost is not found to be a symmetry of the system, even though the condition $\theta^{0i} = 0$ is Galilean invariant. Finally, the Hall conductivity is computed and we find that there is no θ-correction.

Having studied this effective commutative quantum mechanical system upto first order in θ, we set out to enquire whether and how quantum mechanics of noncommutative systems can be carried out for all orders in θ. To that end, we have constructed physically equivalent families of noncommutative Hamiltonians. The implementation of this program to all orders in the noncommutative parameter is carried out in the case of a free particle and harmonic oscillator moving in a constant magnetic field in two dimensions. The role played by the SW map has also been investigated in details. It is found that this spectrum preserving map coincides with the SW map in the absence of interactions, but not in the presence of interactions. Furthermore, a new possible paradigm for noncommutative quantum Hall systems was demonstrated in a simple setting. Here an interacting commutative system is traded for a weakly interacting noncommutative system, resulting in the same physics for the low energy sector. This provides a new rational for the introduction of noncommutativity in quantum Hall systems.

We then present a very simple and elegant approach, which is somewhat complementary to the point of view presented above, to understand the quantum Hall system from the noncommutative framework. The role that interactions play in the noncommutative structure that arises when the relative coordinates of two interacting particles are projected onto the lowest Landau level is discussed in detail. It is shown that the interactions in general renormalize the noncommutative parameter away from the non-interacting value $\frac{1}{B}$. The effective noncommutative parameter is in general also angular momentum dependent. The filling fractions at incompressibilty (which are in general renormalized by the interactions) is obtained by an heuristic argument, based on the noncommutative coordinates. The results are consistent with known results in the case of singular magnetic fields.

We have then also looked at the twisted formulation of noncommutative quantum field theory in the context of NR framework. This is interesting as it has been observed recently that the usual violation of Lorentz symmetry, arising from the non-transforming noncommutative constant matrix $\theta^{\mu\nu}$ in $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$ can be restored through the twisted implementation of Lorentz group *a la* Drin'feld [127]. So the question naturally arises is regarding its status in NR system, where the relevant symmetry group is the Galilean group. Balachandran et.al [114] have shown that for this new twisted action, the Bose and Fermi commutation relations of relativistic field gets deformed as well to render statistics as a super-selected observable. In this thesis, we carry out the NR version of the above analysis. We have shown the twisted Galilean invariance of the noncommutative parameter particularly under rotation, even in presence of spacetime noncommutativity, as we find that the Galileo boost generators become related simply to the linear momentum generators and thereby remain unaffected by twist. We also obtained the deformed algebra of the Schrödinger field in configuration and momentum space by studying the action of the twisted Galilean group on the NR limit of the Klein-Gordon field, which can eventually be extended for a Dirac field as well in a straightforward manner. Using this deformed algebra we compute the two particle correlation function in a canonical ensemble to show that the repulsive statistical potential between a pair of identical (twisted) fermions can saturate to a finite value at coincident points, thereby violating Pauli's exclusion principle. However, it can be clearly seen that any possible effect is not detectable at present energies.

Finally, we would like to mention that the issue of braided twisted symmetry as discussed in [128] has not been investigated in this thesis.

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